Asymptotic Expansions: Their Derivation and Interpretation

R. B. DINGLE

Department of Theoretical Physics,
University of St. Andrews,
Fife, Scotland

1973

ACADEMIC PRESS
LONDON AND NEW YORK
A Subsidiary of Harcourt Brace Jovanovich, Publishers

Chapter I

Asymptotics—A Behavioural Survey

1. Origin and Nature of Asymptotic Expansions

To help fix ideas, let us start by examining types of expansion which may be developed for the well-known error function

$$\phi(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-u^2}\, du. \tag{1}$$

This function is important in its own right, and of especial interest in asymptotics through having provided one of the earliest examples historically of a Stokes discontinuity (Stokes, 1864).

Expansion of the exponential as a power series, followed by term by term integration, leads to the absolutely convergent series

$$\phi(x) = \frac{2x}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-x^2)^s}{s!(2s+1)}\,. \tag{2}$$

Though theoretically exact for all magnitudes and phases of the variable, such a convergent series can prove dismally inconvenient except for small values. For instance, in the series (2), individual terms do not begin to decrease until $s \sim |x^2|$, and their sum does not even begin to approximate the function well until about three times as many terms have been assembled. More seriously, for large $|x|$ the final sum is far smaller than the largest individual terms, which therefore have to be calculated to many extra significant figures. In more advanced examples than (2), the presence within the summation of a factor which is not so simple—e.g. a zeta-function or worse—can render this a daunting task.

Fortunately, the alternative "asymptotic" approach produces a series in which, by contrast, ease of calculation to a prescribed accuracy increases with $|x|$. We shall first deal with the phase sector $|\mathrm{ph}\, x| < \frac{1}{2}\pi$.

The observation $2\pi^{-\frac{1}{2}}\int_0^\infty e^{-u^2}\,du = 1$ enables us to write

$$\phi(x) = 1 - 2\pi^{-\frac{1}{2}}\int_x^\infty e^{-u^2}\,du. \tag{3}$$

In the new integral, e^{-u^2} is of significant magnitude only near the lower limit $u = x$, and can therefore be expanded about this point. It is convenient to choose as expansion parameter the variable $f = u^2 - x^2$, whereupon

$$\phi(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}}\int_0^\infty e^{-f}\left(1 + \frac{f}{x^2}\right)^{-\frac{1}{2}} df. \tag{4}$$

Without pausing at this stage to examine the validity of ensuing steps, the expansion about the point $f = 0$ required for insertion is

$$\left(1 + \frac{f}{x^2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(r-\frac{1}{2})!}{r!}\left(-\frac{f}{x^2}\right)^r, \tag{5}$$

so

$$\phi(x) = 1 - \frac{e^{-x^2}}{x\pi}\sum_0^\infty \frac{(r-\frac{1}{2})!}{r!\,(-x^2)^r}\int_0^\infty e^{-f} f^r\,df$$

$$= 1 - \frac{e^{-x^2}}{x\pi}\sum_0^\infty \frac{(r-\frac{1}{2})!}{(-x^2)^r}, \qquad |\mathrm{ph}\, x| < \tfrac{1}{2}\pi. \tag{6}$$

As will become clear when we examine the general question of definition in Section 5, it is advisable at the outset to make a careful distinction in terminology between:

(i) the right-hand side of the equation (6), "the asymptotic expansion of $\phi(x)$ in this phase range",

(ii) the second contribution, "a component asymptotic series", and

(iii) the summation itself, "an asymptotic power series".

This is also semantically strict, since in such a context "expansion" refers to an algebraic dilatation however composed, whereas the word "series" means more narrowly "an ordered sequence of systematically constructed terms".

The terms in the asymptotic power series $\Sigma(r-\frac{1}{2})!/(-x^2)^r$ behave in a radically different way from those in the convergent series $\Sigma(-x^2)^s/s!(2s+1)$. For moderate or large $|x|$, the terms in the former first progressively decrease in magnitude, then reach a minimum around $r \sim |x^2|$ and thereafter increase; while those in the latter first increase, reach a maximum around $s \sim |x^2|$ and thereafter decrease. Because of the ultimate progressive increase in magnitude of its terms, an asymptotic power series is divergent. Nevertheless, even if only crudely broken off at its least term (thereby retaining only the first few terms), it produces remarkably accurate results, especially for large values of the variable.

Why does such a series derived from the asymptotic approach end by diverging? To answer this, let us examine the steps in the foregoing derivation more closely. The binomial theorem for expanding $(1+f/x^2)^\alpha$ is valid only if *either* the expansion terminates (α a positive integer) *or* $|f/x^2| < 1$. The former is not relevant to $(1+f/x^2)^{-\frac{1}{2}}$ in (5), and yet we see that in the next step (6) we did suppose the binomial expansion to hold not only for f from 0 to x^2, but further from x^2 to ∞. The first terms in (6) yield highly accurate values for $\phi(x)$ when x is large because the exponential factor e^{-f} makes the integral negligible long before f reaches x^2 and the questionable region beyond. But however large x may be, sufficiently late terms diverge because of the extension of the binomial expansion beyond its circle of convergence.

More generally, the asymptotic expansion for a function $\int e^{-F(u)} G(u)\, du$, where G is slowly varying, is ascertained by first separating out the ranges of integration, U say, through which e^{-F} decreases monotonically right down to zero (equation (3) in our example), then within these ranges changing the variable of integration to F, thus

$$\int_U e^{-F}\, G\, du = \int_{F_0}^{\infty} e^{-F}\, [G/(dF/du)]\, dF$$

(cf. (4)), expanding in each $G/(dF/du)$ as a Taylor series (cf. (5)), and integrating term by term (cf. (6)). Each resultant asymptotic series is ultimately divergent because in its progress from F_0 to ∞ the variable F reaches and then exceeds the radius of convergence in the F-plane of this Taylor series for $G/(dF/du)$.

Next, we examine the consequences of outstepping the circle of convergence. No error of magnitude or phase has been incurred; when $|f/x^2| \geqslant 1$ the binomial expansion (5) retains perfect precision of meaning, namely that the series is to be summed [to $(1+f/x^2)^{-\frac{1}{2}}$] in exactly the same way as if it had lain within the circle of convergence. The ultimate convergence failure in an asymptotic power series thus has its origin in a solely-symbolic mechanism of continuation, not involving any numerical inexactitude. Moreover the technical misdemeanour in continuation can be exactly atoned by applying a reverse process of symbolic continuation when interpreting late terms. This conclusion is in stark contrast with the long accepted *non-sequitur* according to which such an expansion must contain an inherent vagueness and inaccuracy because its late terms are not comprehensible as they stand†.

By a theorem of Darboux (1878), late terms in a Taylor series originate from the singularity in the function expanded which lies closest to the origin of expansion (Chapter VII, Section 2). For example, late terms in (5) originate from the branch point of $(1+f/x^2)^{-\frac{1}{2}}$ located at $f=-x^2$. More generally, if α is a positive integer $(1+f/x^2)^\alpha$ possesses no singularities and its expansion correspondingly terminates and so has no late terms; whereas if α is a negative integer so the function has a pole at $f=-x^2$, or if α is fractional so it has a branch point there, the expansion does not terminate and its late terms are dictated by the singularity. According to the binomial theorem,

$$\left(1+\frac{f}{x^2}\right)^\alpha = \frac{1}{(-\alpha-1)!}\sum_0^\infty \frac{(r-\alpha-1)!}{r!}\left(-\frac{f}{x^2}\right)^r, \tag{7}$$

showing the late terms to be alike whether the singularity consists of a pole or a branch point. By the Darboux theorem, late terms in a Taylor series for some more complicated function of f will also be of similar form, since they depend only on the behaviour of that function in the immediate neighbourhood of its singularity closest to the origin of expansion at $f=0$. Reference to the derivation of (6) then leads to the expectation that, barring pathological cases, sufficiently late terms $r \gg 1$ of *any* asymptotic power series will transpire to be expressible in a standard limiting form $(r+\text{constant})!/(\text{variable})^r$, the accuracy of this limiting representation increasing with r. (The full representation for finite r will consist of a decreasing sequence of like contributions). This conclusion, which will be verified in varying contexts throughout our investigation, is critically important in two ways: first, because it provides a valuable lead on how asymptotic power series and expansions containing them might best be defined; and second, because it shows that substantially a single theory of interpretation will apply equally to late terms of all such asymptotic series (Chapters XXI onwards).

† There are indications that some mathematicians, active in the field before the alleged vagueness got written into the theory by Poincaré's prescription, were unsure of this inference. Especially interesting is the reservation in parenthesis in the following extract from Stokes' own account of the Stokes discontinuity: "A semiconvergent series (considered numerically, and apart from its analytic form) defines a function only subject to a certain amount of vagueness which is so much the smaller as the modulus of the variable is larger". Stokes may well have realised that while the sum of an asymptotic series can be determined only approximately from the numerical values of its terms, this does not exclude the possibility that precise information might be extractable from their analytic form.