

CHAOS THEORY



Predrag Cvitanovic'

Chaos and what to do about it
Classics Illustrated version



Navier-Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^i{}_{klm} = \frac{\partial \Gamma^i{}_{km}}{\partial x^l} - \frac{\partial \Gamma^i{}_{kl}}{\partial x^m} + \Gamma^i{}_{ne} \Gamma^n{}_{km} - \Gamma^i{}_{nm} \Gamma^n{}_{ke}$$

Yang-Mills

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_a{}^{\mu\nu}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g C_{abc} A_\mu^b A_\nu^c$$

Quantum?

Newtonian straightjacket

infinitesimal time:

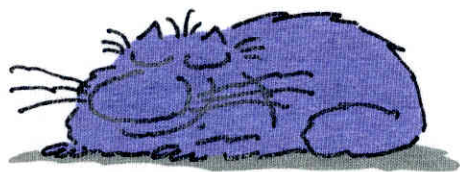
$$\frac{d}{dt} X_i = v_i(x)$$



long time outcome:

$$X_i(t) = \text{no clue}$$

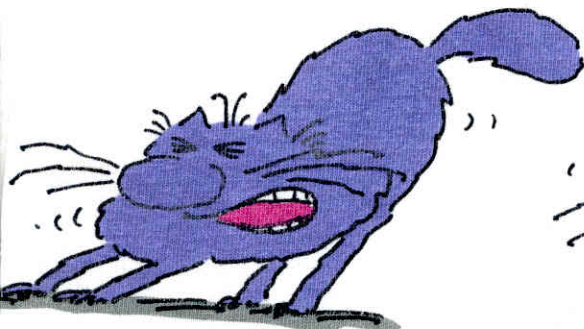
cat aerobics...



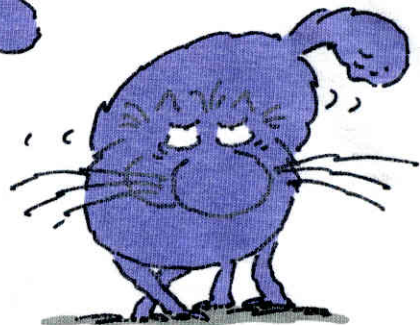
1. Relax



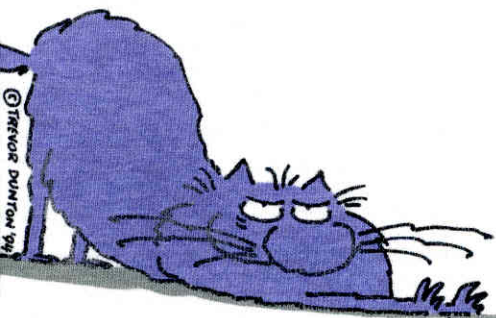
2. Get ready



3. Stretch



4. Twist



5. Bend.



6. And Relax



(Poincaré 1882)

Chaos is everywhere

Chaos = stretch



+
fold



(repeat)ⁿ



CLASSICAL CHAOS

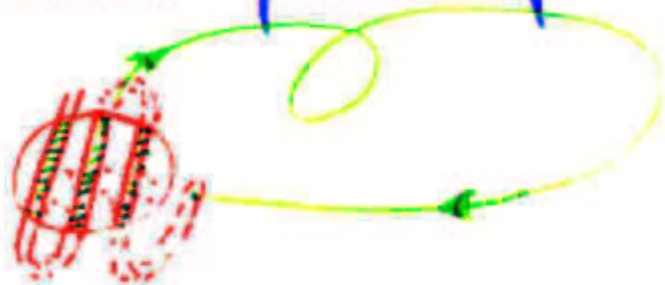
Poincaré (1890) +

1. sensitivity to initial conditions



$$\text{Lyapunov } \frac{1}{T} \ln |\text{stretching factor}| > 0$$

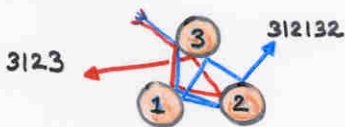
2. bounded phase space



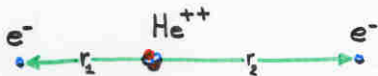
$$\text{entropy } \frac{\ln |\# \text{ pieces}|}{T} > 0$$

EXAMPLES OF CHAOS

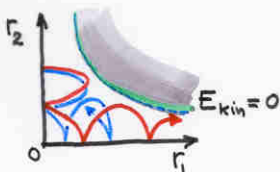
pinball



colinear helium



$$H = \frac{P_1^2}{2} + \frac{P_2^2}{2} - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_1 + r_2}$$



H. Poincaré, describing in
“Les méthodes nouvelles de la mécanique méleste” his discovery
of homoclinic tangles:

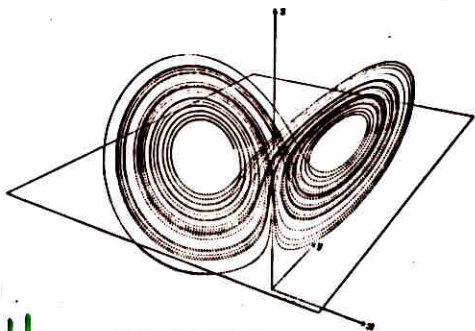
The complexity of this figure will be striking, and I
shall not even try to draw it.

LORENZ 1963

METEOROLOGY

74

О. Ленфорд



'Raynolds
no. "

Рис. 2. $\dot{x} = -10x + 10y$
 $\dot{y} = 28x - y - \frac{8}{3}xy$
 $\dot{z} = -\frac{8}{3}x + xy$

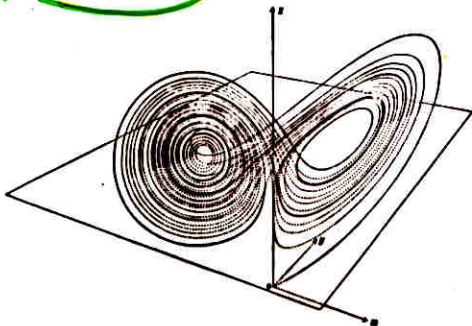
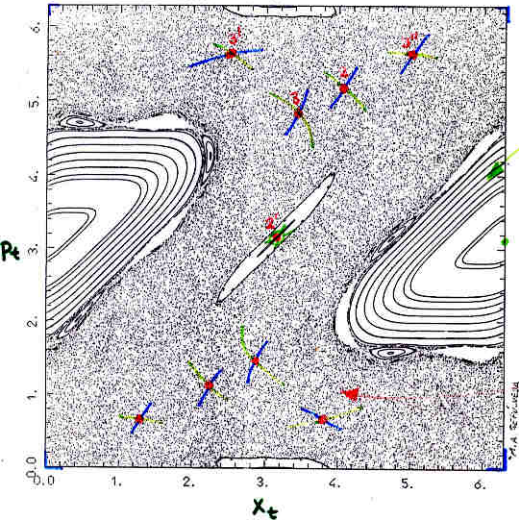


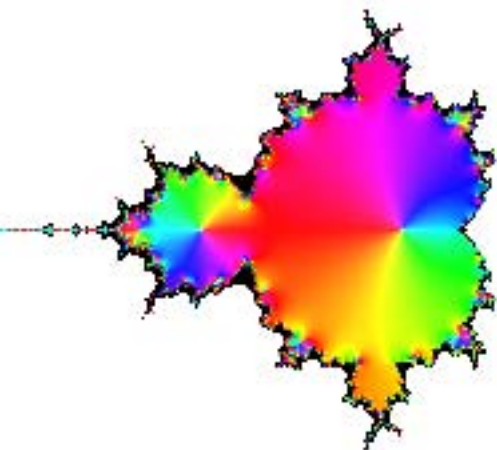
Рис. 3. $\dot{x} = -10x + 10y$
 $\dot{y} = 40x - y - \frac{8}{3}xy$
 $\dot{z} = -\frac{8}{3}x + xy$

WEATHER: UNPREDICTABLE

STANDARD MAP

$$p_{t+1} = p_t + K_f(x_t)$$
$$x_{t+1} = x_t + p_{t+1} \pmod{2\pi}$$

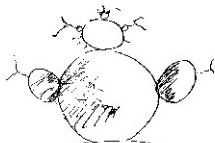
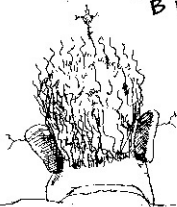






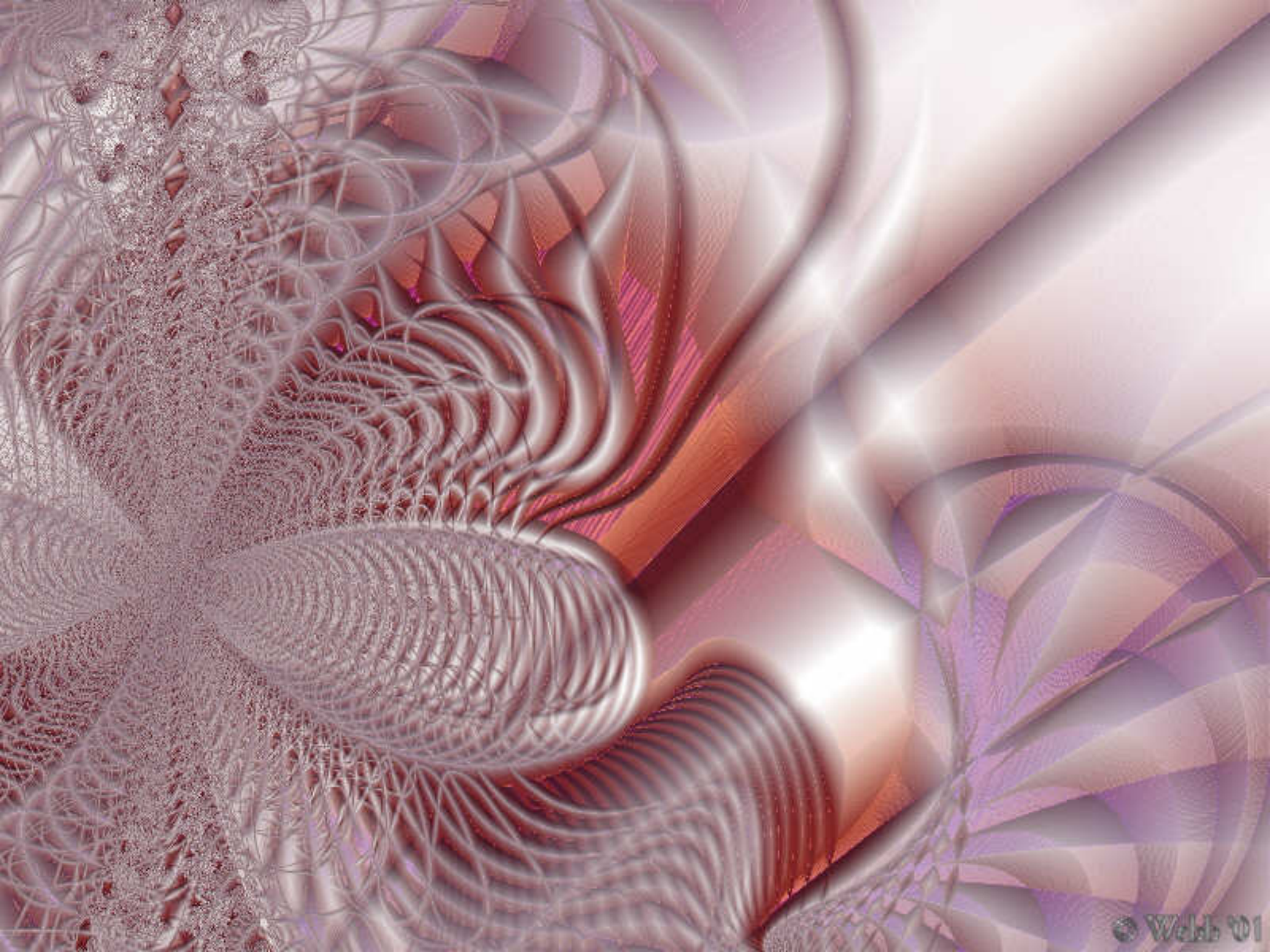
B B Mandelbrot,

June 1984









¿ how quickly we lose
control ?

¿ how many degrees of freedom
are out of control ?

Lyapunov time for chaotic systems

$$\mathbb{I}(t) \Rightarrow \mathbb{I}(t) \text{ most unstable eigenvalue} \\ = 10^{\lambda t}$$

Lyapunov time = $\frac{1}{\lambda}$ = # seconds to lose a digit of accuracy

examples:

Pluto	10^{15} sec
Obliquity of Mars	10^{11} sec
Chemical chaotic oscillator	10^3 sec
Hydrodynamic chaotic flow	10 sec
cm ³ Argon at room temperature	10^{-10} sec
cm ³ Argon at triple point	10^{-15} sec


when

$T_{\text{Lyapunov}} \ll T_{\text{observation}}$

Chaos Rules

what to do ?

(Ruelle '71)



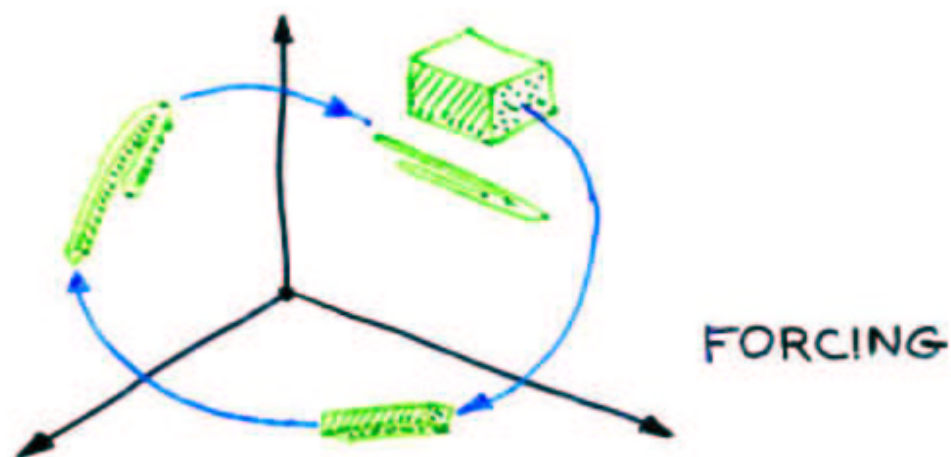
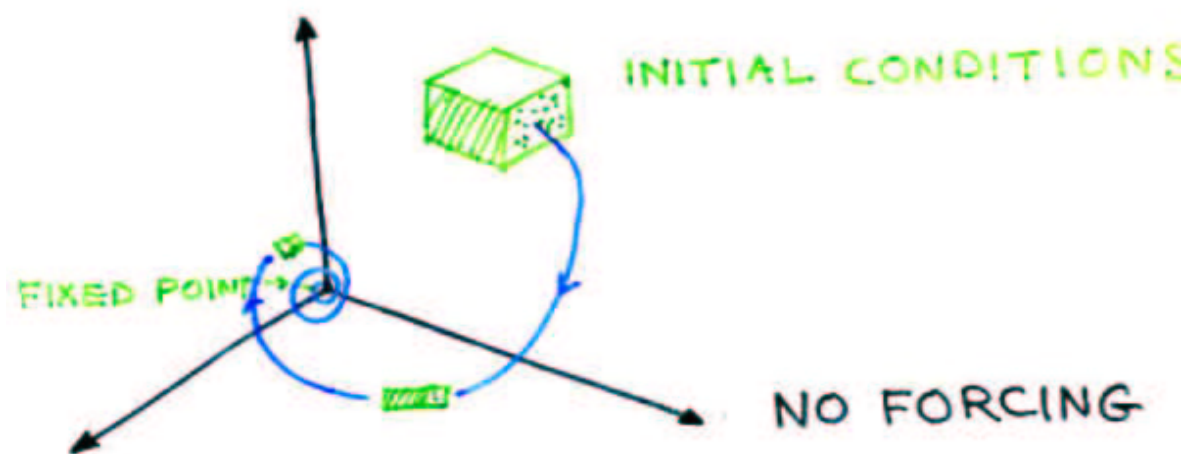
Strange attractors

3 degrees of freedom
suffice for (weak)
turbulence

(do not need zillions of
parts to make world complicated)

DISSIPATION

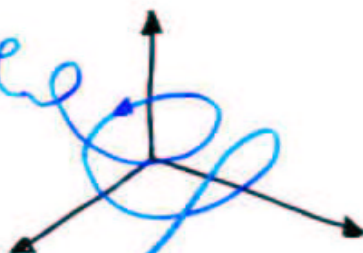
⇒ LOW DIMENSIONAL
ATTRACTOR



DYNAMICAL SYSTEM



PHASE SPACE



DISSIPATION



LOW DIMENSIONAL ATTRACTOR



POINCARÉ SECTION



1-DIMENSIONAL ITERATIVE MODELS



MODE LOCKINGS,
PERIOD DOUBLINGS,

WHAT IS THE PROBLEM ?

"CHAOS" = ϵ 

⇒ DETAILED PREDICTION
IMPOSSIBLE

BUT!:

"DETERMINISTIC" =

ϵ_0 

⇒ ASYMPTOTIC GEOMETRY
RIGID, ORDERED

HENCE THE GOALS OF "DETERMINISTIC CHAOS" ARE

1. DESCRIBE GEOMETRY
2. EVALUATE MEASURABLE AVERAGES
 $\langle \varphi \rangle$, $\langle \varphi(t) \varphi(0) \rangle$ CORRELATIONS, ...

WHY IS THIS
DIFFERENT FROM THE
TRADITIONAL PHYSICS?

GIVE UP ON:

INITIAL CONDITIONS + $\frac{d\vec{x}}{dt} = \vec{f}(x, \dots)$

⇒ DESCRIPTION OF
THE SYSTEM

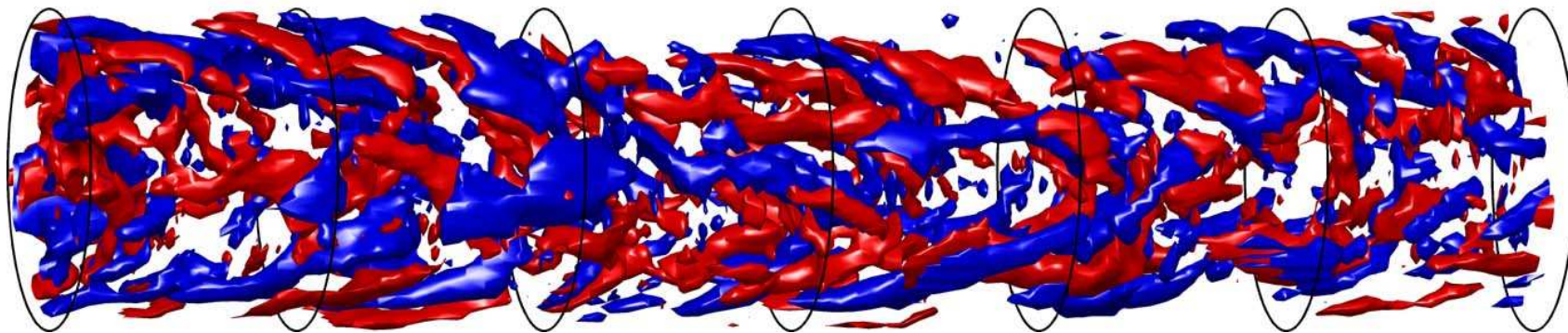
INSTEAD:

EXTREMELY NON-LINEAR
PHYSICS DESCRIBED BY
SELF-SIMILARITY EQUATIONS!



New experiments: Unstable Coherent Structures

Stereoscopic Particle Image Velocimetry → 3-d velocity field over the entire pipe¹



Observed structures resemble numerically computed traveling waves

What lies beyond?



chaos **classical and quantum**

available at: ChaosBook.org
cover design: P. Cvitanovic
"Clouds over Croatia"

Predrag Cvitanović - Roberto Artuso
Per Daintyist - Ronnie Mainieri
Gregor Tanner - Sándor Vattay
Niall Whelan - Andreas Wirtz

Future looks bright

