19.20 The volume of fluid carried along by the flow between cylinders of length L per unit of time becomes

$$Q = \int_a^b v_\phi(r)\, L dr = \frac{1}{2} L a^2 \Omega \left(\frac{b^2}{b^2 - a^2} \log \frac{b^2}{a^2} - 1 \right) \,. \qquad \text{(19-A5)}$$

This is equivalent to the volumetric discharge rate for pipe flow, although no fluid is actually discharged here.

19.21 Per unit of length the kinetic energy is

$$\frac{T}{L} = \int_0^r v_\phi^2\, 2\pi r\, dr = 2\pi \frac{a^4 b^2 \Omega^2}{b^2 - a^2} \left(\frac{1}{4} \frac{a^2}{b^2} - \frac{3}{4} + \frac{b^2}{b^2 - a^2} \log \frac{b}{a} \right)$$

19.22 In cylindrical coordinates assume that the flow field is radial, $\boldsymbol{v} = v_r(r)\, \boldsymbol{e}_r$ outside the pipe. Volume conservation implies that $v_r 2\pi r L$ is the same for all r. Hence $v_r(r) = Q/2\pi r$ where Q is the volume flow through the pipe wall per unit of pipe length.

19.25 a) $Q = adL\Omega/2 \approx 470\mathsf{cm}^3/\mathsf{s}$. b) $\mathcal{E} = 2\pi\eta\Omega^2 a^3 L/d \approx 10\mathsf{J/s}$.

19.26 Using (18-18) we obtain

$$\begin{aligned} P &= -2\eta \int_V \sum_{ij} v_{ij}^2 = -\eta \int_V (\nabla_r v_z)^2 \\ &= -\eta \int_0^a \left(-\frac{G}{2\eta} r \right)^2 2\pi r L dr = -\frac{\pi G^2 a^4 L}{8\eta} \\ &= -QGL \end{aligned}$$

20 Creeping flow

20.2 Let $\boldsymbol{v}$ be the velocity field in the rest frame of the body. The total work of the body on the fluid is in the rest frame of the asymptotic fluid

$$\oint_{\mathcal{S}} \sum_{ij} (v_i - U_i)(-\sigma_{ij})\, dS_j = \sum_i U_i \oint_{\mathcal{S}} \sum_j \sigma_{ij}\, dS_j = \boldsymbol{U} \cdot \boldsymbol{\mathcal{F}} = U\mathcal{D} \qquad \text{(20-A1)}$$

where it is used that $v_i = 0$ at the surface of the body.

20.3 a) Follows from linearity of the field equations and the pressure independence of the boundary conditions.

b) The field equations are

$$\boldsymbol{\nabla}^2 R_{ij} = \boldsymbol{\nabla}_i Q_j \qquad \text{(20-A2)}$$

$$\sum_i \nabla_i R_{ij} = 0 \qquad \text{(20-A3)}$$

The boundary conditions are for $|\boldsymbol{x}| \to \infty$

$$R_{ij}(\boldsymbol{x}) \to \delta_{ij} \qquad \text{(20-A4)}$$

$$Q_i(\boldsymbol{x}) \to 0 \qquad \text{(20-A5)}$$

At the surface of the body the velocity field must vanish, $\boldsymbol{n} \cdot \mathbf{R}(\boldsymbol{x}) = 0$.

c) The stress tensor is

$$\sigma_{ij} = -p\delta_{ij} + \eta(\nabla_i v_j + \nabla_j v_i) = \eta \sum_k \tau_{ijk} U_k \tag{20-A6}$$

where

$$\tau_{ijk} = -\delta_{ij} Q_k + \nabla_i R_{jk} + \nabla_j R_{ik} \tag{20-A7}$$

The total force on the body with surface $\mathcal{S}$ is

$$\mathcal{F}_i = \oint_{\mathcal{S}} \sum_j \sigma_{ij} dS_j = \eta \sum_k U_k \oint_{\mathcal{S}} \tau_{ijk} dS_j \tag{20-A8}$$

This shows that

$$S_{ik} = \oint_{\mathcal{S}} \tau_{ijk} dS_j \tag{20-A9}$$

may be understood as a form factor, such that

$$\mathcal{F}_i = \eta \sum_k S_{ik} U_k \tag{20-A10}$$

20.4 a) The discharge is at $\theta = \pi/2$

$$Q = \int_a^b (-v_\theta) 2\pi r \, dr = \pi (b-a)^2 \left(1 + \frac{a}{2b}\right) U$$

b) The ratio is

$$\frac{Q}{\pi(b^2 - a^2)U} = \frac{b-a}{a+b}\left(1 + \frac{a}{2b}\right)$$

c) The ratio vanishes because of the no-slip condition which requires the velocity to vanish at the surface of the sphere.

20.6

(a) Write $\boldsymbol{x} = r\boldsymbol{e}_r$ and use (C-15) to obtain

$$\frac{d\boldsymbol{x}}{dt} = \frac{dr}{dt}\boldsymbol{e}_r + r\frac{d\theta}{dt}\boldsymbol{e}_\theta$$

(b) Combine the differential equations to obtain

$$\frac{d\theta}{dr} = -\frac{B}{A}\tan\theta$$

which is a solvable first order equation. The integral over r is carried out by means of partial fractions

$$\frac{B}{A} = \frac{r^2 + \frac{1}{4}ar + \frac{1}{4}a^2}{r(r-a)(r+\frac{1}{2}a)} = -\frac{1}{2r} + \frac{1}{r-a} + \frac{1}{2r+a}$$

(c) For $r \to \infty$ we get $d \to r\sin\theta = \sqrt{x^2 + y^2}$.

(d) Put $\theta = \frac{\pi}{2}$ to get $d = (r-a)\sqrt{1 + a/2r}$ where r is the point of closest approach.

