17

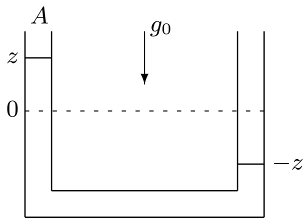
Global laws of balance

Momentum, angular momentum, and kinetic energy, are purely mechanical quantities that like mass are carried along with the movement of material. A jet of water carries momentum, and when the jet hits a wall it exerts a force which may be calculated from the amount of momentum it deposits per unit of time. Besides the mechanical quantities, there are thermodynamic quantities such as internal energy and entropy, also transported along with the material. The mechanical and thermodynamic quantities mentioned here are all *extensive*, meaning that the amount in a composite body is the sum of the amounts in the parts. For continuous matter, where body parts may be of any size down to material particles, an extensive quantity is always described by a density field, and the amount of the quantity in a volume is calculated as the integral over the density. Other so-called *intensive* quantities, for example pressure and temperature, are not additive like the extensive quantities.

We have already seen that mass is transported without actual loss or gain, as expressed by the local and global equations for mass conservation. Extensive quantities are generally not conserved, but have *sources* that create and destroy them. Forces create momentum, moments create angular momentum, work creates kinetic energy. The global laws of balance for extensive quantities basically state the obvious: *the amount of a quantity produced by the sources in a volmume is either accumulated in the volume or leaves the volume through the surface.*

Global laws of balance often provide the first coarse estimate of the behavior of fluids. Many ordinary fluids are effectively incompressible, and furthermore so close to ideal that internal friction (*i.e.* viscosity) can be disregarded in the first approximation. Balancing the budgets for mass, momentum, angular momentum, and kinetic energy in such fluids is relatively easy and yields valuable information about their dynamics without without having to solve the field equations. In this chapter we shall mainly apply the global mechanical laws of balance to ideal fluids that do not conduct heat. Heat will be discussed in chapter 28.

17.1 Connected tubes



When the water is at rest, the water level will be the same in connected vertical tubes. The tube may have any shape between the two vertical sections.

We shall begin with a simple example of the use of energy conservation in a fluid system to derive how water moves in one of the most basic experiments in fluid dynamics. Consider a long straight tube with cross section A which is bent through 180° somewhere in the middle and placed with the open sections vertically upwards. Water is filled into the system, and as everybody knows, gravity will eventually make the levels of water equal in the two vertical tubes. Before reaching equilibrium the water sloshes back and forth with diminishing amplitude. Basic physics knowledge tells us that the energy originally given to the water oscillates between being kinetic and potential, while slowly draining away because of internal friction in the water.

Even if we do not know the exact solution to the fluid flow problem, we are nevertheless able to make a reasonable quantitative estimate of the behavior of the water. When the water level in one vertical tube is raised by z relative to the equilibrium level, mass conservation tells us that it is lowered by the same amount in the other vertical tube. The average velocity of the water in one tube is taken to be $v = dz/dt$, and the full length of the water column L. The total mass of the water column is $\rho_0 AL$, so that the kinetic energy becomes $\frac{1}{2}\rho_0 ALv^2$. Since a small water column of height z and weight $g_0\rho_0 Az$ effectively has been moved from one vertical tube to the other and thereby raised by z relative to equilibrium, the potential energy becomes $g_0\rho_0 Az^2$. Adding these contributions we get the following estimate for the total energy,

$$\mathcal{E} = \frac{1}{2}\rho_0 ALv^2 + g_0\rho_0 Az^2 \ , \tag{17-1}$$

where ρ_0 is the constant density of water.

For simplicity we assume that there is no friction eat away the energy. The total energy must therefore be conserved, so that its time-derivative has to vanish,

$$\frac{d\mathcal{E}}{dt} = \rho_0 ALv\frac{dv}{dt} + 2g_0\rho_0 Az\frac{dz}{dt} = 0 \ . \tag{17-2}$$

Using that $v = dz/dt$, we find the differential equation for the harmonic oscillator,

$$\frac{d^2z}{dt^2} = -\frac{2g_0}{L}z. \tag{17-3}$$

Had we included friction, there would also have been a damping term on the right hand side. The solution to the harmonic equation is of the form

$$z = a\cos\omega t \tag{17-4}$$

with amplitude a and angular frequency $\omega = \sqrt{2g_0/L}$. This is the frequency of a mathematical pendulum with length $L/2$, but contrary to the pendulum the motion of the water is purely harmonic.

17.2 Overview of the global laws

Before going into the complete discussion of the global quantities and their laws of balance, we shall here outline the contents of these laws. We shall see that in continuum mechanics, the global laws of balance take nearly the same form as the global laws of Newtonian particle mechanics (see appendix B).

There are four global mechanical quantities at play in any physical system, continuous or discrete: 1) the total mass M, 2) the total momentum $\boldsymbol{\mathcal{P}}$, 3) the total angular momentum $\boldsymbol{\mathcal{L}}$, and 4) the kinetic energy $\mathcal{T}$. In continuum mechanics, the amount of any of these quantities in a given volume is defined by an integral over the corresponding density. The global laws obeyed by these quantities are,

$$\frac{DM}{Dt} = 0 \,, \qquad \text{mass conservation} \tag{17-5a}$$

$$\frac{D\boldsymbol{\mathcal{P}}}{Dt} = \boldsymbol{\mathcal{F}} \,, \qquad \text{momentum balance} \tag{17-5b}$$

$$\frac{D\boldsymbol{\mathcal{L}}}{Dt} = \boldsymbol{\mathcal{M}} \,, \qquad \text{angular momentum balance} \tag{17-5c}$$

$$\frac{D\mathcal{T}}{Dt} = P \,. \qquad \text{kinetic energy balance} \tag{17-5d}$$

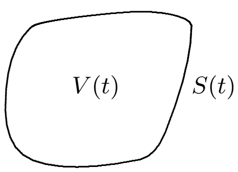
On the left hand side of any of these equations we find the material derivative of the quantity, representing the rate of change of the quantity in a comoving volume. The special notation is necessary in continuum physics where matter is generally permitted to move in and out of the chosen volume. In Newtonian particle mechanics, the number of particles is assumed to be constant, and there appears only the ordinary time derivative.

Each of the laws of balance (17-5) thus expresses that the material rate of change equals the *global source* on the right hand side. The first equation expresses that there is no source of mass, implying that the total mass must be constant in any volume moving along with the flow. The second equation expresses that the source of momentum is the total force $\boldsymbol{\mathcal{F}}$ acting on the material, the third that the source of angular momentum is the total moment $\boldsymbol{\mathcal{M}}$ of all the forces, and the fourth that the source of kinetic energy is the total power P (*i.e.* rate of work) of all forces. By adding in the potential energy in, say, a gravitational field, the last equation may be turned into the equation of balance for the total mechanical energy $\mathcal{E}$, intuitively used in the preceding section. In chapter 28 we shall include heat, and learn how general energy balance is related to the First Law of Thermodynamics.

The global laws all follow from the general equations of motion for continuous matter (15-35), and are for this reason automatically fulfilled for any solution to the field equations. Having found such a solution, you need not to worry about the balance of mass, momentum, angular momentum, or kinetic energy. But when you can't solve the field equations, these laws of balance put useful constraints on the approximations that can be made.

17.3 The control volume

In Newtonian particle mechanics, a "body" is understood as a collection of a fixed number of particles. In continuum mechanics the notion of a body is much more general: any volume — usually called a *control volume* — may be viewed as a "body" at a given time. Intuitively we think of bodies as made from different materials, but the surface of the control volume does not have to correspond to an interface between materials with different properties, although it often is convenient to choose it to coincide with such an interface. In the course of time the control volume may be moved around and deformed any way we desire; this is really why it is called a *control* volume.



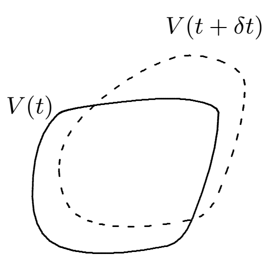
A "body" consists of all the matter contained in an arbitrary time dependent control volume $V(t)$ with surface $S(t)$.

One may wonder whether it is really necessary to consider bodies that general. Previously we have, for example in the discussion of mass conservation, only considered arbitrary fixed volumes which do not change with time. A quick review of the preceding example of connected tubes reveals, however, that the control volume we instinctively used there encompasses all the water in the system and moved along with it. Not permitting moving control volumes would in fact put unreasonable restraints on our freedom to analyze the physics of continuous systems.

Reynolds transport theorem

Before proceeding to a discussion of the global laws of balance for mass, momentum, angular momentum, and kinetic energy, it is necessary to establish some relations for general control volumes. Although the following analysis will be carried through for the total mass in a control volume,

$$M(t) = \int_{V(t)} \rho(\boldsymbol{x}, t)\, dV \ , \tag{17-6}$$



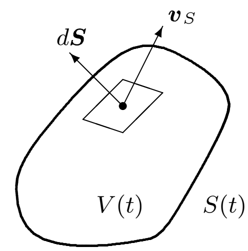
A moving volume changes with time. The change $dV(t)$ is the (signed) volume between the dashed and fully drawn surface outlines.

it is equally valid for any other extensive quantity if the mass density is replaced with the density of the quantity in question.

The time dependence of the total mass has two origins: the changing density and the moving control volume. To calculate the rate of change of the mass at time t, we consider the control volume a small time interval δt later where it has changed to $V(t+\delta t) = V(t) + dV(t)$. Expanding to first order in the small quantities, the change in mass of the control volume is

$$\begin{aligned} M(t+\delta t) - M(t) &= \int_{V(t+\delta t)} \rho(\boldsymbol{x}, t+\delta t)\, dV - \int_{V(t)} \rho(\boldsymbol{x}, t)\, dV \\ &\approx \delta t \int_{V(t)} \frac{\partial \rho(\boldsymbol{x}, t)}{\partial t}\, dV + \int_{dV(t)} \rho(\boldsymbol{x}, t)\, dV \ . \end{aligned}$$

The volume integral over the little change in volume $dV(t)$ only gets contributions near the surface $S(t)$ of the control volume. Let us introduce the velocity $\boldsymbol{v}_S(\boldsymbol{x}, t)$ of a surface element $d\boldsymbol{S}$ in the point $\boldsymbol{x}$ at time t of $S(t)$. This velocity needs only



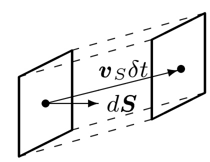
Every surface element $d\boldsymbol{S}$ of the control volume may move with a different velocity $\boldsymbol{v}_S(\boldsymbol{x}, t)$.

be defined on the surface itself and not all over space, so it is not a field in the usual sense of the word. Since each surface element $d\boldsymbol{S}$ of the moving surface scoops up a (signed) volume $dV = \boldsymbol{v}_S \delta t \cdot d\boldsymbol{S}$ in a small time interval δt, we find

$$\int_{dV(t)} \rho(\boldsymbol{x}, t)\, dV \approx \delta t \oint_{S(t)} \rho(\boldsymbol{x}, t)\, \boldsymbol{v}_S(\boldsymbol{x}, t) \cdot d\boldsymbol{S} \; .$$

Suppressing the explicit dependence on space and time, the rate of change of mass in the moving control volume, $dM/dt = \lim_{\delta t \to 0}(M(t + \delta t) - M(t))/\delta t$, becomes,

$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t}\, dV + \oint_S \rho\, \boldsymbol{v}_S \cdot d\boldsymbol{S} \; . \tag{17-7}$$

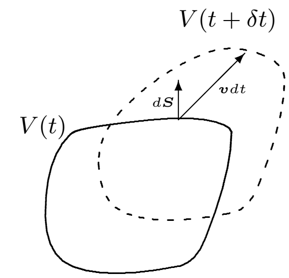


A surface element $d\boldsymbol{S}$ scoops up the volume $\boldsymbol{v}_S\delta t \cdot d\boldsymbol{S}$ in a small time interval δt.

The first term is the contribution from local change in density, and the second is the signed amount of mass which the surface of the moving control volume incorporates per unit of time.

If the surface of the control volume is always constant in time, $\boldsymbol{v}_S = \boldsymbol{0}$, the control volume is said to be *fixed*. If on the other hand the surface of the control volume always follows the material, $\boldsymbol{v}_S = \boldsymbol{v}$, it is said to be *comoving* (in which case it is actually not under our control!). We shall as before use the symbol D/Dt for the *material time derivative*, defined as the rate of change of mass in a comoving volume, and find from (17-7) with $\boldsymbol{v}_S = \boldsymbol{v}$,

$$\frac{DM}{Dt} = \int_V \frac{\partial \rho}{\partial t}\, dV + \oint_S \rho\, \boldsymbol{v} \cdot d\boldsymbol{S} \; . \tag{17-8}$$

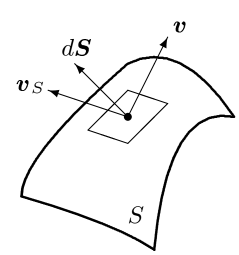


The surface of a comoving volume follows the flow of matter.

It is important to notice that this expression does not depend on the surface velocity $\boldsymbol{v}_S$, implying it may be used to calculate the instantaneous material time derivative for any control volume V with surface S at any time t from a knowledge of the density $\rho(\boldsymbol{x}, t)$ and the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. How the control volume actually moves does not matter.

Combining the material rate of change (17-8) with the general expression (17-7) for the rate of change, we may write

$$\boxed{\frac{DM}{Dt} = \frac{dM}{dt} + \oint_S \rho\, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S} \; .} \tag{17-9}$$



Matter moves with velocity $\boldsymbol{v}$ through the surface element $d\boldsymbol{S}$ which itself moves with velocity $\boldsymbol{v}_S$. The velocity of the matter relative to the moving surface is $\boldsymbol{v} - \boldsymbol{v}_S$.

Since the matter moves with velocity $\boldsymbol{v} - \boldsymbol{v}_S$ relative to the control surface, the surface integral represent the net rate of loss of mass through the moving surface.

This equation goes under the name of *Reynolds' transport theorem* (1903) and is the general basis for global analysis in continuum physics. Generalizing to an arbitrary quantity, it has a fairly clear intuitive content: *the material rate of change of an extensive quantity in an arbitrary control volume equals the actual rate of change of the quantity in the control volume plus its net rate of loss through the surface.*

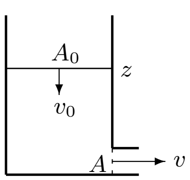
17.4 Mass balance

Mass conservation denotes the claim that mass cannot be created or destroyed, but only moved around. We have previously derived the global equation of mass conservation (15-24) and comparing with (17-8) we find that the material derivative of the total mass in a control volume must vanish,

$$\boxed{\frac{DM}{Dt} = 0 \ .} \tag{17-10}$$

It is in fact rather obvious that the mass of a comoving control volume must be constant, since by definition no mass flows through its surface. This equation is also formally called the *global equation of mass balance* although in this case there are no source terms on the right hand side to balance the material rate of change.

Example 17.4.1: If you choose a control volume equal to your own body surface, it is to a good approximation comoving and your mass is constant, whichever way you move around while holding your breath. When you breathe, the control volume is no more perfectly comoving, and its mass will change with a couple of grams per second in tune with the flux of air through the surface where it covers your open mouth.



The water leaves the container through the drain.

Example 17.4.2: Suppose a cistern filled with water is flushed through a short drain pipe in the bottom of the container. For simplicity we assume that the drain pipe has cross section A, and that the cistern is vertical with cross section A_0. We choose a control volume encompassing all the water in the cistern, so that if the water level is z, the total mass of the water is $M = \rho_0 A_0 z$. The rate of loss through the drain is $\rho_0 A v$, where v is the average velocity of the drain flow. Global mass balance (17-10) and Reynolds theorem (17-9) now leads to,

$$\frac{DM}{Dt} = \frac{dM}{dt} + \rho_0 A v = \rho_0 A_0 \frac{dz}{dt} + \rho_0 A v = 0 \ . \tag{17-11}$$

But since $dz/dt = -v_0$ where v_0 is the average downwards velocity of the water in the cistern, this becomes $A_0 v_0 = A v$ which is nothing but Leonardo's law (15-13).

17.5 Momentum balance

The total momentum in a control volume is obtained by integrating the momentum density $\rho \boldsymbol{v}$ over the volume,

$$\boldsymbol{\mathcal{P}} = \int_V \rho \, \boldsymbol{v} \, dV \ . \tag{17-12}$$

Again it should be emphasized that we may choose the control volume to contain any amount of matter and let it change with time in any way we wish.

Global equation of momentum balance

Replacing ρ by ρv_i, we find from the material derivative (17-7),

$$\frac{D\mathcal{P}_i}{Dt} = \int_V \frac{\partial(\rho v_i)}{\partial t}\, dV + \oint_S \rho v_i\, \boldsymbol{v} \cdot d\boldsymbol{S} = \int_V \left(\frac{\partial(\rho v_i)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho v_i \boldsymbol{v}) \right) dV \ ,$$

where we in the last step have used Gauss' theorem (4-22) to convert the surface integral. The integrand may be further simplified by means of the continuity equation (15-25)

$$\begin{aligned} \frac{\partial(\rho v_i)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho v_i \boldsymbol{v}) &= v_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial t} + v_i \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) + \rho (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v_i \\ &= \rho \left(\frac{\partial v_i}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) v_i \right) \ . \end{aligned}$$

Expressing the parenthesis in terms of the material derivative (15-30), and going back to vector form, we have obtained the relation,

$$\frac{D\boldsymbol{\mathcal{P}}}{Dt} = \int_V \rho \frac{D\boldsymbol{v}}{Dt}\, dV \tag{17-13}$$

It is clear that a similar relation will hold for any other quantity for which the density is proportional to the mass density.

Finally, using the fundamental equation of continuum dynamics (15-35), *i.e.* $\rho D\boldsymbol{v}/Dt = \boldsymbol{f}^*$, the *global equation of momentum balance* becomes,

$$\boxed{\frac{D\boldsymbol{\mathcal{P}}}{Dt} = \boldsymbol{\mathcal{F}} \ ,} \tag{17-14}$$

where the source on the right hand side is the total force (9-16) acting on the control volume,

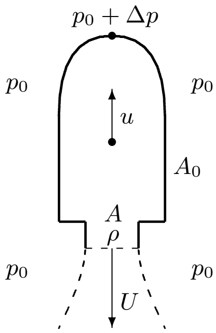
$$\boxed{\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}^*\, dV = \int_V \boldsymbol{f}\, dV + \oint_S \boldsymbol{\sigma} \cdot d\boldsymbol{S} \ .} \tag{17-15}$$

If the volume forces obey Newton's third law, internal forces will not contribute to the first integral (see appendix B).

The material rate of change of momentum may similarly be calculated from Reynolds theorem (17-9) by replacing the mass density ρ by the momentum density ρv_i. In vector form Reynolds theorem for momentum becomes,

$$\boxed{\frac{D\boldsymbol{\mathcal{P}}}{Dt} = \frac{d\boldsymbol{\mathcal{P}}}{dt} + \oint_S \rho \boldsymbol{v}\, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S} \ .} \tag{17-16}$$

Now we may read the equation of momentum balance (17-14) from right to left as saying that *the momentum produced by volume and contact forces acting on the material in the control volume is either accumulated in the control volume ($d\boldsymbol{\mathcal{P}}/dt$) or discharged through its surface.*



Sketch of a rocket moving upwards with vertical velocity u while emitting material of density ρ and relative velocity U through an opening of cross section A. The control volume moves with the rocket and follows the outside of the hull, cutting across the exhaust opening. There is ambient air pressure p_0 everywhere around the rocket, also at the nozzle outlet. The small rise in pressure Δp at the front of the rocket is the main cause of drag.

Launch of a small rocket

A rocket accelerates upwards by burning chemical fuel and spewing the hot reaction gases downwards. The hot gases are emitted from the rocket through an opening (nozzle) with cross section A, and we assume for simplicity that the density of the gas ρ and its velocity U relative to the rocket remain unchanged during the burn. We also assume that the gas velocity is much smaller than the velocity of sound, which it is in toy rockets, so that the pressure at the exit may be taken to be equal to the ambient atmospheric pressure p_0.

In this case it is most convenient to choose a control volume which follows the *outside* of the rocket, cutting across the nozzle outlet. Such a control volume moves everywhere with the instantaneous speed u of the rocket. It contains at any moment all the material of the rocket, including the fuel and the burning gases, but not the gases that have been exhausted through the nozzle. The rate of loss of mass through the nozzle, $Q \approx \rho U A$, is constant by assumption, implying that the rocket mass must decrease linearly from its initial value M_0 at $t = 0$, so that its mass at time t is,

$$M = M_0 - Q\,t \; . \tag{17-17}$$

At the end of the burn, when all the fuel has been spent, a "payload" mass $M_1 < M_0$ remains. The time it takes to burn the fuel mass $M_0 - M_1$ is $t_1 = (M_0 - M_1)/Q$. After the burn the rocket flies ballistically like a cannon ball, subject only to the forces of gravity and air resistance. Here we are only interested in establishing the equation of motion valid from liftoff to burnout.

The absolute vertical velocity (relative to the ground) of the exhaust gases is $v = u - U$, allowing us to estimate the momentum loss integral in (17-16) as $\rho v U A = (u - U)Q$. Assuming that the center of gravity remains fixed relative to the rocket (and that is by no means sure), the total momentum is $\mathcal{P} = Mu$, and the material derivative (17-16) becomes

$$\frac{D(Mu)}{Dt} = \frac{d(Mu)}{dt} + (u - U)Q = M\frac{du}{dt} - UQ \; . \tag{17-18}$$

In the last step we have used the expression (17-17) for the rocket mass.

The total vertical force on the rocket is the sum of its weight $-Mg_0$ and the resistance or drag caused by the interaction of the rocket's hull with the air. Air drag has two components: *skin drag* from viscous friction between air and hull, and *form drag* from the changes in pressure at the hull caused by the rocket "punching" through the atmosphere. Form drag can for example be estimated from the Bernoulli stagnation pressure increase at the tip of the rocket, $\Delta p \approx \frac{1}{2}\rho_0 u^2$, where ρ_0 is the density of air. Multiplying with the cross section of the rocket A_0, the estimate of the form drag becomes $-\frac{1}{2}\rho_0 u^2 A_0$ (with opposite sign if u is negative). Form drag thus grows quadratically with rocket speed, and at high velocity it dominates skin drag which only grows linearly with velocity. Leaving out skin drag, the total vertical force on the control volume may be

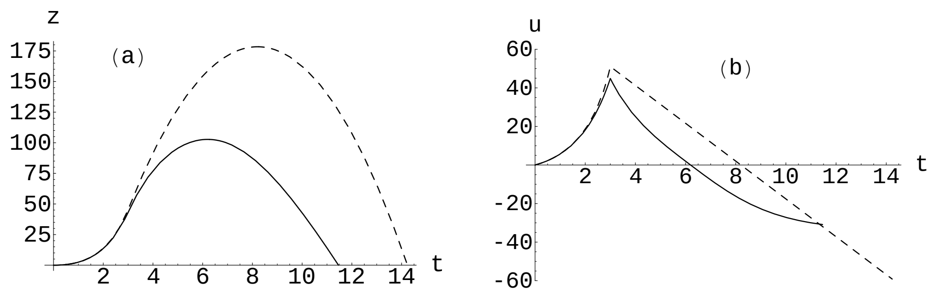


Figure 17.1: *Height (a) and velocity (b) as a function of time during vertical flight of a small fireworks rocket with form drag (solid) and without (dashed). The cusp in the velocity happens at the end of the burn and signals the transition to ballistic flight. The rocket has diameter* 6 cm*, total mass* 1 kg*, payload* 0.2 kg*, and form drag coefficient* $C_D = 1$*. The rocket burns for* 3 s*, emitting gases at a speed of* 50 m/s*. The numeric solution of the rocket equation (17-20) shows that with form drag the rocket reaches a velocity of* 45 m/s *at the end of the burn and a height of* 43 m*. During the subsequent ballistic flight, it reaches a maximum height of* 103 m *about* 6 s *after start, and finally it falls back and hits the ground again with a speed of* 30 m/s *about* 11.5 s *after start. Without form drag the rocket would reach a maximum height of* 178 m*, and it would hit the ground with a speed of* 60 m/s *after 14 seconds.*

written

$$\mathcal{F} = -Mg_0 - \frac{1}{2}\rho_0 u^2 A_0 C_D \ . \tag{17-19}$$

where we have included a dimensionless factor C_D, called the *drag coefficient*, which takes into account the actual shape of the rocket.

Equating the material derivative (17-18) with the total force (17-19), and dividing by the mass (17-17), we find the "rocket equation" for vertical flight during the burn,

$$\boxed{\frac{du}{dt} = -g_0 + \frac{UQ - \frac{1}{2}\rho_0 u^2 A_0 C_D}{M_0 - Qt}} \ . \tag{17-20}$$

This shows that the rocket will take off from rest $u = 0$ at $t = 0$ provided the initial acceleration is positive, *i.e.* $UQ > M_0 g_0$. The differential equation can only be solved numerically (except in vacuum for $C_D = 0$ where it can be solved analytically; see problem 17.2), and the results are shown for a typical fireworks rocket in fig. 17.1 with $C_D = 1$. The figure also includes the period of ballistic flight which follows after the burn and brings the rocket to a maximal height before it turns around and falls back to earth. One notices how the form drag reduces the maximum height for this rocket to a little more than half of what it would be in vacuum (dashed curve). If the rocket shape were made highly streamlined, the form drag coefficient could be made considerably smaller than unity, allowing it to attain greater heights for the same amount of fuel (see page 554 for a general discussion of form drag).

17.6 Reaction forces

A rocket accelerating in vacuum while spewing hot gases in the opposite direction is at first sight rather mysterious, because there are apparently no external forces that can account for the acceleration. On reflection we understand that the hig-speed gas streaming out of the rocket nozzle carries negative momentum away from the rocket and thus adds positive momentum to the rocket itself (in the direction of flight). Formally, this may be exposed by rewriting momentum balance (17-14) with the left hand side given by Reynolds theorem (17-16) as

$$\frac{d\boldsymbol{\mathcal{P}}}{dt} = \boldsymbol{\mathcal{F}} - \oint_S \rho \boldsymbol{v} \, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S} \; . \tag{17-21}$$

The integral (including the sign) represents the net momentum flux *into* the control volume through its surface. Although this term simulates the action of an external force, it must be emphasized that it is not a true external force but rather a "fictitious" force akin to the centrifugal force.

In particular, it is not a reaction force arising as a response to the action of another force in accordance with Newton's third law. But why are rockets and jet aircraft then said to be "reaction-driven"? We shall come back to this question after a discussion of momentum balance in steady flow.

Total force in steady flow

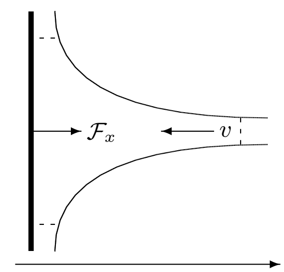
Fluids flowing through tubes, pipes, ducts, and conduits are important components in many of the machines used in the home, in transport and in industry. Everywhere in such a system the conduit walls will act on the fluid with certain contact forces, and by Newton's third law the fluid must respond with equal and opposite reaction forces. This definition is purely a matter of convention; there is no deeper distinction between the agents in an action/reaction pair. One might as well say that the conduit walls react to the forces exerted by the fluid. To prevent confusion it is often wiser to avoid using the concept of reaction force, or at least make clear in each case what is reacting to what, and where.

In steady flow, it is natural to choose a fixed control volume. Since the flow pattern remains the same for all time, the total momentum of the material in the fixed control volume will be constant, $d\boldsymbol{\mathcal{P}}/dt = \boldsymbol{0}$. From momentum balance (17-21) it follows that the total external force on the control volume must be,

$$\boxed{\boldsymbol{\mathcal{F}} = \oint_S \rho \boldsymbol{v} \, \boldsymbol{v} \cdot d\boldsymbol{S} \; .} \tag{17-22}$$

The total force is still given by the sum of the volume and contact forces (17-15), so this relation should be seen as a constraint which necessarily must be satisfied by these forces to secure the steady flow. Contrary to hydrostatics ($\boldsymbol{v} = \boldsymbol{0}$) where the total force on any volume has to vanish, the environment is required to exert a generally non-vanishing total force on the control volume, determined by the flux of momentum *out* of the control volume. The total force is unambiguous and may, as shown by the integral, always be calculated or estimated from the flow through the openings where fluid is let in and out of the control volume.

Example 17.6.1 (Water cannon): A steady jet of water hits a fixed wall and produces no back-splash in the process, but just streams symmetrically away over the wall (when gravity is ignored). A fixed control volume is chosen which contains all of the water in the jet with inlet far from the wall and the outlet far from the inlet. The force normal to the wall (in the x-direction) is obtained from the momentum flux at the inlet because the outflow is approximatively parallel with the wall. Let the inlet area be A and let the average inlet velocity be of magnitude v, so that the total (outgoing) volume flux through the inlet becomes $-vA$. The momentum density normal to the wall is similarly $-\rho_0 v$, leading to the estimate of the total normal force,



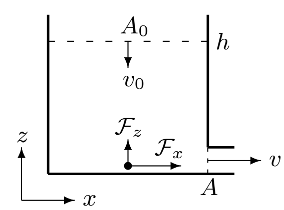
A jet of water hitting a wall. The control volume is limited by the jet's surface and by the dashed lines.

$$\mathcal{F}_x \approx \rho_0 v^2 A \ . \tag{17-23}$$

This force can only come from the wall and may be viewed as the wall's reaction to the impact of the incoming fluid.

Police use water cannon to control crowds. With a jet diameter of 1.5 inch and volume flux 20 liter per second, the water speed becomes about 18 m/s and the jet reaction to about 350 N, corresponding to the weight of 35 kg. You don't stay on your feet for long after being hit by such a jet, although it will probably not hurt you seriously.

Example 17.6.2 (Draining cistern): A cistern with cross section A_0 drains steadily with velocity v through a spout of cross section $A \ll A_0$ while being constantly refilled to maintain a water level h above the spout. Assuming nearly ideal flow, we may use the Toricelli result (16-20) that the outlet speed equals the free-fall speed from height h, *i.e.* $v \approx \sqrt{2g_0 h}$. In the steady state, the required horizontal force on the water is,



Draining cistern. The loss of momentum from the water leaving the container through the drain requires a horizontal reaction force $\mathcal{F}_x$ from the cistern. There is also a much smaller vertical reaction force F_z.

$$\mathcal{F}_x \approx \rho_0 v^2 A \ . \tag{17-24}$$

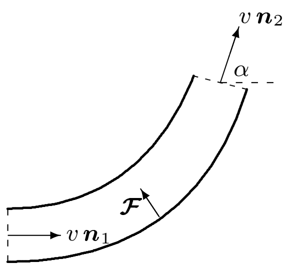
This force must be ultimately provided by the static friction between the cistern and the floor that supports it.

The water moving slowly down through the cistern with vertical velocity $v_0 = vA/A_0$ requires a much smaller force,

$$\mathcal{F}_z \approx \rho_0 v_0^2 A_0 = \frac{A}{A_0}\mathcal{F}_x \ . \tag{17-25}$$

As for the water cannon, the positive sign can be understood from the fact that the water transports a negative mass flux $-v_0 A_0$ along z with a negative momentum density $-\rho_0 v_0$. This force must also be provided by the floor in addition to the weight of the cistern plus water.

With the parameters of the wine barrel (example 16.3.2), *i.e.* $A \approx 20$ cm^2, $A_0 \approx 0.8$ m^2, and $h \approx 2$ m, one finds $\mathcal{F}_x \approx 77$ N, corresponding to the weight of almost eight liters of wine. The vertical force becomes $\mathcal{F}_z = 0.2$ N which only amounts to the weight of a small thimbleful of wine.

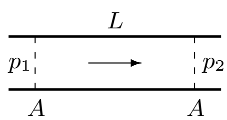


A pipe bending through an angle α.

Example 17.6.3 (Bend in a pipe): A pipe with cross section A bends through an angle α. The control volume is chosen to contain all the fluid between inlet and outlet. Incompressible fluid flows steadily through the pipe with velocity v, leading to a total force

$$\boldsymbol{\mathcal{F}} \approx (\boldsymbol{n}_2 - \boldsymbol{n}_1)\rho_0 v^2 A \ , \tag{17-26}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the inlet and outlet directions of the flow. The magnitude of the force is $\mathcal{F} = |\boldsymbol{n}_2 - \boldsymbol{n}_1|\,\rho_0 v^2 A = 2(1-\cos\alpha)\rho_0 v^2 A$. The force derives partly from the interior walls of the pipe and partly from the pressures at inlet and outlet. Thus, a 90° bend in a 1 inch pipe with water flowing at 3 m/s requires a total force of magnitude 9 N.



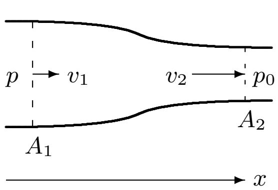
Incompressible fluid passes steadily through a section of a pipe with a pressure drop between inlet and outlet. The only force which can balance the pressure force on the fluid is friction due viscous shear stress acting at the inner surface of the pipe.

Example 17.6.4 (Viscous stress): Incompressible fluid is sent through a section of a straight pipe with inner radius a and length L. The inlet pressure p_1 is higher than the outlet pressure p_2. The control volume contains all the fluid in the pipe section. Since inflow equals outflow, the required total force on the fluid is zero, $\mathcal{F}_x = 0$. The total force is composed of the pressure forces at the ends of the pipe and shear stress σ due to friction at the inner surface,

$$\mathcal{F}_x = p_1 \pi a^2 - p_2 \pi a^2 - \sigma 2\pi a L = 0 \tag{17-27}$$

Solving for σ we find $\sigma = \Delta p\, a/2L$ where $\Delta p = p_1 - p_2$ is the pressure drop along the pipe. Thus, if the pressure in a half inch water pipe is measured to drop 140 Pa per meter, the shear stress must be $\sigma \approx 0.9$ Pa.

Nozzle puzzle



A nozzle formed by narrowing down a pipe from the inlet A_1 to the open outlet A_2.

Whereas it is always unambiguous what the required total force must be for a fixed control volume in steady flow, it can be quite hard in practice to decide how a piece of equipment will act on the external supports that keep it in place.

As an illustration we consider a nozzle — the metal device sitting at the end of a firehose — formed by narrowing down the cross section of a straight pipe from A_1 to A_2 over a fairly short distance. Mass conservation guarantees that the same amount of water flows in and out

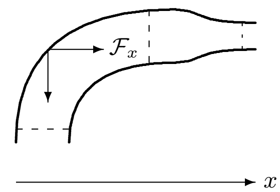
$$Q = \rho_0 v_1 A_1 = \rho_0 v_2 A_2 \ , \tag{17-28}$$

and the total force in the downstream direction becomes

$$\mathcal{F}_x = \rho_0 v_2^2 A_2 - \rho_0 v_1^2 A_1 = \frac{Q^2}{\rho_0}\left(\frac{1}{A_2} - \frac{1}{A_1}\right) \ . \tag{17-29}$$

The required force is positive (for $A_2 < A_1$) and it follows that the fireman, as expected, must push forward on the nozzle.

For $A_1 = A_2$ the total force vanishes, and that seems to contradict the common gardening experience that it is always necessary to hold firmly on to an open hose (even without nozzle) because it otherwise will flail back and drench you. So what is the actual extra force that must be applied by the hands that hold a nozzle when water is pouring out? The puzzle is resolved when one realizes that a soft but unstretchable hose always meanders through bends and turns before the nozzle. Consider, for example, a firehose with an unsupported 90° bend somewhere before the nozzle. Including the bend in the control volume, the required total force along the nozzle is obtained from (17-26) and equals the rate of loss of momentum through the outlet without any contribution from the inlet,



A nozzled firehose with an unsupported 90° bend. The extra force required by the bend is transmitted down from the nozzle through the soft but unstretchable material of the hose.

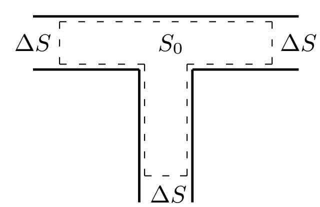
$$\mathcal{F}_x = \rho_0 v_2^2 A_2 = \frac{Q^2}{\rho_0 A_2} \; . \tag{17-30}$$

What happens here is that the unsupported bend demands a total force from the environment which in this case is transmitted down to the bend from the nozzle through the unstretchable material of the hose. It is definitely better to use this value for the purpose of calculating the required strength of the firefighter.

Example 17.6.5: Older fire hoses with 2.5 inch diameter equipped with a 1.5 inch nozzle can typically deliver 20–40 liters of water per second. At a rate of 20 liters per second, the required nozzle force (with a bend) is 350 N which is just about manageable for a single firefighter. A modern 5 inch fire hose can deliver up to 100 liters of water per second through a 3 inch nozzle, leading to a required force of about 2200 N. This nozzle cannot even be handled by three firefighters, but should be firmly anchored in a boat or truck.

Formal definition of reaction force

In most systems, fluid flows in and out of the control volume through well-defined openings which together constitute a piece ΔS of the control volume surface S. Let S_0 be the remainder so that the closed surface is composed of two open parts, $S = S_0 + \Delta S$. Here we shall choose the surface S_0 to follow the inside of the fluid conduits but it might also have been chosen to follow the outside of the system, as we did for the rocket on page 298.



A control volume for a T-junction containing all the fluid between the inner conduit walls S_0 and the inlets and outlets ΔS.

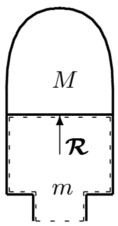
The *absolute reaction force* is defined to be the resultant of all the contact forces that the contents of the control volume exerts on S_0,

$$\boldsymbol{\mathcal{R}} = -\int_{S_0} \boldsymbol{\sigma} \cdot d\boldsymbol{S} \; . \tag{17-31}$$

The surface contribution to the total force (17-15) may now be split into the reaction force plus a part from the openings ΔS, and from momentum balance (17-21) we obtain an exact expression for the reaction force,

$$\boxed{\boldsymbol{\mathcal{R}} = -\frac{d\boldsymbol{\mathcal{P}}}{dt} + \int_V \boldsymbol{f}\, dV + \int_{\Delta S} \boldsymbol{\sigma} \cdot d\boldsymbol{S} - \int_{\Delta S} \rho \boldsymbol{v}\, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S} \; .} \tag{17-32}$$

The last term represents the net momentum flux *into* the control volume. Although it in itself is not a true reaction force, momentum balance makes it appear as a contribution to the reaction force.



The control volume encircles the chamber in the rocket where the fuel burns. The reaction force $\boldsymbol{\mathcal{R}}$ is the total force exerted on the walls by the gas in the chamber. It is this reaction force which lifts the solid structure of the rocket.

Finally, we are able to clear up the question of why rockets and jets are said to be *reaction-driven*. Choosing the control volume to encompass only the chamber in which the gases burn, the reaction force that these gases exert on the walls of the chamber is given by the above expression. In this case, the first two terms are tiny because they are proportional to the mass m of the burning gasses, which is normally much smaller than the mass M of the rocket. The third term can also be disregarded because there is essentially no stress in the exhaust (at least at subsonic speeds). Thus, for the rocket or jet engine, the reaction force may to a very good approximation be calculated from the last term, representing the net gain of momentum from the gases expelled through the exhaust nozzle.

Several simplifications are possible in the above expression for the reaction force. In steady flow with a fixed control volume, the first term vanishes, $d\boldsymbol{\mathcal{P}}/dt = \mathbf{0}$. Usually the only volume force is constant gravity $\boldsymbol{g} = (0, 0, -g_0)$, making the volume integral equal to the weight $M\boldsymbol{g}$ where M is the total mass of the control volume. Although there may be shear stresses at play in the openings ΔS, they tend to be small and to cancel because of symmetry so that the stress tensor may be replaced by the pressure $-p$ in the third term (even for viscous flow). With these simplifications the steady-flow reaction force on a fixed control volume may be written,

$$\boxed{\boldsymbol{\mathcal{R}} = M\boldsymbol{g} - \int_{\Delta S} p\, d\boldsymbol{S} - \int_{\Delta S} \rho \boldsymbol{v}\, \boldsymbol{v} \cdot d\boldsymbol{S}\ .} \tag{17-33}$$

The first term is the well-known weight reaction which for incompressible flow is independent of the velocity and thus equal to the static weight of the fluid. For compressible flow the mass of the fluid could depend on the velocity of the flow; if you, for example, increase the speed of the gas in a natural gas pipeline by increasing the inlet pressure, the gas in the pipeline becomes denser and thus weighs more.

Role of the ambient atmosphere

Most machines are immersed in an ambient "atmosphere" of air (or water), which for simplicity is assumed to be at rest with a hydrostatic pressure distribution p_0. If the fixed control volume were filled with ambient fluid at rest, the absolute (and static) reaction force would be,

$$\boldsymbol{\mathcal{R}}_0 = M_0\boldsymbol{g} - \int_{\Delta S} p_0\, d\boldsymbol{S}\ , \tag{17-34}$$

where M_0 is the mass of the ambient fluid. Usually, we are interested in the change $\boldsymbol{\Delta} R = \boldsymbol{\mathcal{R}} - \boldsymbol{\mathcal{R}}_0$ in the reaction force that occurs when the ambient fluid is replaced with the correct fluid that is intended to fill the system. This *relative reaction force* becomes,

$$\boxed{\Delta\boldsymbol{\mathcal{R}} = (M - M_0)\boldsymbol{g} - \int_{\Delta S} (p - p_0)\, d\boldsymbol{S} - \int_{\Delta S} \rho \boldsymbol{v}\, \boldsymbol{v} \cdot d\boldsymbol{S}\ .} \tag{17-35}$$

Evidently, the relative reaction force is obtained by reducing the mass of the fluid by the buoyancy of the displaced ambient fluid and calculating all pressures relative to the ambient pressure.

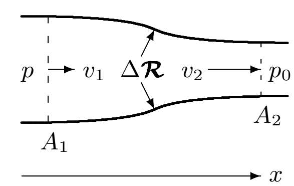
The difference $p - p_0$ between absolute pressure and the ambient pressure is called the *gauge* (or *gage*) pressure, because that is what you would read on a manometer gauge.

Example 17.6.6: A natural gas pipeline at the bottom of the sea transports gas between two islands. During construction the engineers allowed sea water to fill the pipeline which rested comfortably at the bottom under its own weight. But when the system was filled with gas, the buoyancy of the displaced water made the pipeline float to the surface. The engineers got fired for failing to correctly calculate the relative reaction force.

Nozzle puzzle, continued

We are now in position to calculate the relative reaction force from the firehose nozzle discussed on page 302. Disregarding gravity and taking the outlet pressure equal to the ambient pressure p_0, the relative reaction force becomes

$$\Delta\mathcal{R}_x = (p - p_0)A_1 - \rho_0 v_2^2 A_2 + \rho_0 v_1^2 A_1 \ . \qquad (17\text{-}36)$$



The true reaction force on a nozzle points along the direction of flow because there is a higher pressure before and in the constriction than at the outlet.

For nearly ideal flow, Bernoulli's theorem allows us to find the pressure difference between inlet and outlet,

$$\frac{1}{2}v_1^2 + \frac{p}{\rho_0} = \frac{1}{2}v_2^2 + \frac{p_0}{\rho_0} \ . \qquad (17\text{-}37)$$

Together with mass conservation (17-28), everything can be expressed in terms of the inlet and outlet areas, and the resultant may be written

$$\Delta\mathcal{R}_x = \frac{(A_1 - A_2)^2}{2A_1A_2}\,\frac{Q^2}{\rho_0 A_2} \ . \qquad (17\text{-}38)$$

Somewhat surprisingly the fluid exerts a positive reaction force on the nozzle along the direction of flow. On reflection this is in fact in agreement with expectations because the pressure on the inside of the nozzle constriction is larger than the ambient pressure surrounding the nozzle.

The reaction force $\Delta\mathcal{R}_x$ is the force that the fireman would have to oppose, should the firehose suddenly break. As we have seen, this is quite different from the force that must normally be applied to keep the nozzle and unbroken hose in place.

Example 17.6.7: For the older firehose in example 17.6.5, the required handle force changes instantaneously from +350 N to −200 N when the hose breaks. Since these forces are rather large, there is a real risk that the fireman will fly off together with the nozzle.

17.7 Angular momentum balance

The *angular momentum* (sometimes called the moment of momentum) of a material particle with momentum $d\boldsymbol{\mathcal{P}}$ is

$$d\boldsymbol{\mathcal{L}} = \boldsymbol{x} \times d\boldsymbol{\mathcal{P}} = \rho\, \boldsymbol{x} \times \boldsymbol{v}\, dV \ , \tag{17-39}$$

and the total angular momentum of the material in a control volume V is,

$$\boldsymbol{\mathcal{L}} = \int_V \rho\, \boldsymbol{x} \times \boldsymbol{v}\, dV \ . \tag{17-40}$$

Angular momentum depends on the origin of the coordinate system. If we shift the origin by $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$, the angular momentum shifts by $\boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}} + \boldsymbol{a} \times \boldsymbol{\mathcal{P}}$, unless the total momentum vanishes which is generally not the case in fluid systems.

Although this definition of angular momentum seems self-evident, there are some subtle assumptions behind. The main worry is that the above expression only contains the angular momentum of the center-of-mass motion of the material particles. The fast random (thermal) motion of the molecules of a gas could in principle also contribute, but as long as there is no correlation between the position and velocity of the random component of molecular motion (and why should there be?), this contribution will average out (see problem 17.10).

Global equation of angular momentum balance

The material derivative of the angular momentum is calculated in the same way as for the momentum (17-13) with the result,

$$\frac{D\boldsymbol{\mathcal{L}}}{Dt} = \int_V \rho \frac{D(\boldsymbol{x} \times \boldsymbol{v})}{Dt}\, dV \tag{17-41}$$

A single material time derivative of a field acts like a normal time derivative with respect to products. Using that $D\boldsymbol{x}/Dt = \boldsymbol{v}$ and $\rho D\boldsymbol{v}/Dt = \boldsymbol{f}^*$, we find

$$\rho \frac{D(\boldsymbol{x} \times \boldsymbol{v})}{Dt} = \rho \frac{D\boldsymbol{x}}{Dt} \times \boldsymbol{v} + \rho \boldsymbol{x} \times \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{x} \times \boldsymbol{f}^* \ .$$

Thus, the *global equation of angular momentum balance* takes the form,

$$\boxed{\frac{D\boldsymbol{\mathcal{L}}}{Dt} = \boldsymbol{\mathcal{M}} \ ,} \tag{17-42}$$

where

$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \boldsymbol{f}^*\, dV \ , \tag{17-43}$$

is the total moment of the forces acting on the material particles in the control volume. This quantity also depends on the origin of the coordinate system, and transforms as $\boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}} + \boldsymbol{a} \times \boldsymbol{\mathcal{F}}$ under a shift $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$. Both sides of angular momentum balance (17-42) thus shift in the same way so that the form of this relation is unchanged.

Reynolds theorem for angular momentum is obtained by replacing ρ with $\rho\,(\boldsymbol{x} \times \boldsymbol{v})_i$ in (17-9), and becomes on vector form,

$$\boxed{\frac{D\boldsymbol{\mathcal{L}}}{Dt} = \frac{d\boldsymbol{\mathcal{L}}}{dt} + \oint_S \rho\, \boldsymbol{x} \times \boldsymbol{v}\, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S}\ .} \tag{17-44}$$

As for momentum, this is in practice the most useful formulation.

It is a bit more complicated to derive the global form of the total moment than it was for momentum. Inserting the effective force (9-18), we find

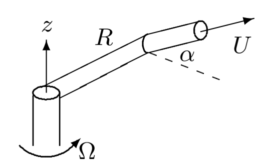
$$(\boldsymbol{x} \times \boldsymbol{f}^*)_i - (\boldsymbol{x} \times \boldsymbol{f})_i = \sum_{jkl} \epsilon_{ijk} x_j \nabla_l \sigma_{kl} = \sum_{jkl} \epsilon_{ijk} \nabla_l (x_j \sigma_{kl}) - \sum_{jk} \epsilon_{ijk} \sigma_{kj}\ .$$

The first term on the right hand side is a divergence which after integration leads to a surface integral. The last term vanishes when the stress tensor is symmetric, which we assume it is. In complete analogy to the total force (9-16), the total moment of force may thus be written

$$\boxed{\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \boldsymbol{f}\, dV + \oint_S \boldsymbol{x} \times \boldsymbol{\sigma} \cdot d\boldsymbol{S}\ .} \tag{17-45}$$

If the volume forces are central, which is the case for gravity, internal forces do not contribute to the first term (see appendix B).

Spinning up a rotating lawn sprinkler



Arm of a lawn sprinkler. The nozzle is at right angles to the arm and has elevation angle α against the azimuthal direction. The control volume envelops the rotating sprinkler and cuts across the nozzle outlet and the pivot inlet.

Some rotating lawn sprinklers are constructed with two horizontal arms of length R mounted on a common pivot. Each arm is a tube carrying water from the pivot towards a nozzle that is bent by 90° with respect to the tube and elevated through an angle α against the azimuthal direction. The nozzle outlet has cross section A and the water emerges with velocity U relative to the nozzle. When the water is turned on, the sprinkler starts to rotate and reaches after a while a steady situation in which it rotates with constant angular velocity, determined by the friction moment from the pivot. We are here interested in analyzing the spin-up of the sprinkler when the water pressure is first turned on.

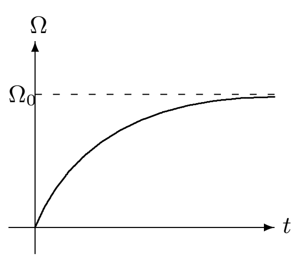
We choose a control volume which follows the outer surface of the sprinkler, cutting across the nozzle outlets and horizontally through the pivot. In cylindrical coordinates with the z-axis along the axis of rotation, the angular momentum of the sprinkler around the z-axis is $\mathcal{L}_z = I\,\Omega$ where I is its moment of inertia of the whole sprinkler and $\Omega > 0$ is the instantaneous angular velocity. The moment of inertia should be calculated in the usual way from the mass distribution in the arms, the pivot and the water contained in the system. For incompressible water the moment of inertia is time independent.

The water entering vertically through the pivot carries no angular momentum, so the pivot does not contribute to the surface integral in (17-44), which is entirely given by the water leaving through the nozzle. In cylindrical coordinates we have $(\boldsymbol{x} \times \boldsymbol{v})_z = r v_\phi$, and since the azimuthal velocity of the water emerging from the nozzle is $v_\phi \approx R\Omega - U\cos\alpha$, the material derivative of $\mathcal{L}_z$ becomes (taking both arms into account)

$$\frac{D\mathcal{L}_z}{Dt} \approx \frac{d(I\Omega)}{dt} + 2\rho_0 R(R\Omega - U\cos\alpha)UA \ . \tag{17-46}$$

Disregarding air resistance, the only moment of force acting on the sprinkler arises from the (deliberately imposed) friction in the pivot. Since the friction moment must be negative, $\mathcal{M}_z = -N$, angular momentum balance becomes

$$I\frac{d\Omega}{dt} + R(R\Omega - U\cos\alpha)Q = -N \ . \tag{17-47}$$



Spin-up of the lawn sprinkler.

where $Q = 2\rho_0 UA$ is the total mass flow rate through the system. With initial condition $\Omega = 0$ at $t = 0$, and assuming that the friction moment is constant, the solution to this linear differential equation is,

$$\Omega = \frac{RUQ\cos\alpha - N}{R^2Q}\left(1 - e^{-\lambda t}\right) \tag{17-48}$$

with time constant

$$\lambda = \frac{R^2Q}{I} \ . \tag{17-49}$$

Evidently, the condition for the angular velocity to be positive is that $RUQ\cos\alpha > N$, and the steady state angular velocity is

$$\Omega_0 = \frac{RUQ\cos\alpha - N}{R^2Q} \ , \tag{17-50}$$

independently of the moment of inertia.

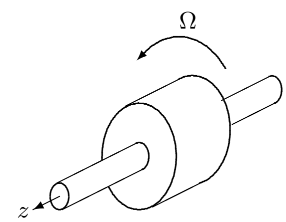
Example 17.7.1 (Lawn sprinkler): A lawn sprinkler with radius $R = 20$ cm, nozzle outlet diameter $d = 4$ mm, and nozzle elevation $\alpha = 30^\circ$, is designed to pass $Q = 0.25$ liter water per second through its two arms and rotate once every 2 seconds in the steady state. The nozzle area becomes $A = \frac{1}{4}\pi d^2 \approx 12.5$ mm^2, the outlet velocity $U = Q/2A\rho_0 \approx 10$ m/s, and the steady state angular velocity $\Omega_0 = \pi$ s^{-1}. The required friction moment that the pivot must deliver, $N = RUQ\cos\alpha - R^2Q\Omega_0 = 0.4$ N m. The horizontal velocity of the water relative to the lawn is $v_\phi = R\Omega_0 - U\cos\alpha \approx -8$ m/s, and the vertical velocity of the water is $v_z = U\sin\alpha \approx 5$ m/s. Assuming a ballistic orbit for the water flying across the lawn (disregarding air resistance) the diameter of the sprinkled region comes to about $D \approx 4\,|v_\phi|\,v_z/g_0 \approx 16$ m. The sprinkled area, $A_0 = \frac{1}{4}\pi D^2 \approx 200$ m^2, receives about $Q/\rho_0 A_0 \approx 4$ mm rain per hour from the sprinkler.

17.8 Reaction moments

The rotating lawn sprinkler spins up and gains angular momentum by expelling water from the nozzles. This is clearly exposed by writing angular momentum balance (17-42) in the form,

$$\frac{d\boldsymbol{\mathcal{L}}}{dt} = \boldsymbol{\mathcal{M}} - \oint_S \rho\, \boldsymbol{x} \times \boldsymbol{v}\, (\boldsymbol{v} - \boldsymbol{v}_S) \cdot d\boldsymbol{S}\ , \tag{17-51}$$

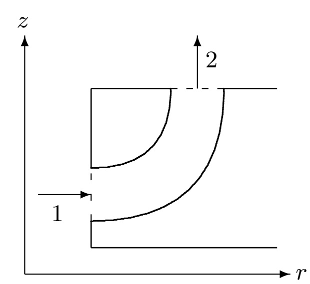
where the last term (including the sign) represents the net flux of angular momentum *into* the control volume through its surface. As for momentum, it must be emphasized that this term is *not* a true moment of force acting on the system, nor the reaction to another moment of force, but a "fictitious" moment of the same nature as the moment created by the centrifugal force in a rotating frame of reference (which will topple you if you try to stand up in a carousel). A closer analysis along the same lines as the reaction force (17-32) will however reveal that angular momentum balance makes this term appear as a contribution to the true reaction moment exerted by the fluid on the conduit walls.



Sketch of a rotor on its shaft, rotating steadily with angular velocity Ω around the z-axis.

Rotor physics

Lawn sprinklers, fans, windmills, hairdryers, propellers, pumps, water turbines, compressors, and jet engines are all examples of the *turbomachinery* that is used ubiquitously in modern homes and industry. In such machines there is a solid rotating part, generically called the *rotor*, which diverts the flow and either puts energy into the fluid or takes it away. The rotor is usually mounted on a rigid shaft, and the energy is transmitted to or from the rotor through an external moment of force, or *torque*, applied to the shaft. The rotor is carefully designed with a number of channels that guide the flow in and out, and mostly the channels are constructed from thin blades that obstruct the flow as little as possible. Rotors essentially only differ by the way their channels are designed. In an *axial flow* rotor, all the channels run along the axis, and in a *radial flow* rotor the channels are all radial. Rotor design is a highly evolved engineering discipline and most real turbomachines employ in fact a mixture of axial and radial flow.



A rotor channel with radial inflow at 1 and axial outflow at 2.

Even if the rotor spins steadily, the flow in the rotor will strictly speaking be unsteady seen from the inertial "laboratory" frame because of the moving blades. In the rotating frame of the rotor, the flow will however be steady, barring the turbulence that may arise if the flow is too rapid. It is for this reason natural to choose a rotating control volume which is fixed with respect to the rotor and contains all the fluid in the rotor. When the rotor spins with angular velocity Ω around the z-axis, the control volume rotates with velocity, $\boldsymbol{v}_S = \Omega\, r\, \boldsymbol{e}_\phi$ (in cylindrical coordinates with z-axis along the rotor axis). The flow velocity relative to the rotor is denoted

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_S = \boldsymbol{v} - \Omega\, r\, \boldsymbol{e}_\phi \tag{17-52}$$

where $\boldsymbol{v}$ is the "absolute" flow velocity in the non-rotating laboratory frame.

In steady operation the total angular momentum contained in the control volume must be constant, $d\boldsymbol{\mathcal{L}}/dt = \mathbf{0}$, in spite of the fact that it rotates. Using that the density of angular momentum along the cylinder axis is $(\boldsymbol{x} \times \boldsymbol{v})_z = r v_\phi$, the total axial moment of force which has to be exerted by the environment on the fluid in the rotor is obtained from angular momentum balance (17-51),

$$\mathcal{M}_z = \oint_S r v_\phi \, \rho \, \boldsymbol{u} \cdot d\boldsymbol{S} \ . \tag{17-53}$$

The environment may contribute in several ways to this moment, for example through friction as we saw for the lawn sprinkler, but if friction can be disregarded, $\mathcal{M}_z$ will be the moment that must be applied to the shaft to keep to rotor spinning steadily. In the following we shall for simplicity assume this to be the case.

The total rate of work, or *power*, that the environment must supply to or take away from the solid shaft is obtained by multiplying the moment $\mathcal{M}_z$ with the angular velocity Ω (see problem 17.5),

$$P = \Omega \mathcal{M}_z \ . \tag{17-54}$$

If the shaft power is positive, the rotor takes energy from the shaft which is put into the flow, as for example in a hair dryer. If on the other hand the shaft power is negative, the rotor produces energy and acts like a turbine by taking energy out of the flow.

Turbomachines running full of fluid, such as a lawn sprinklers, are driven by pressure differences between inlets and outlets. Taking only into account the pseudo-gravitational potential $-\frac{1}{2}\Omega^2 r^2$ of the centrifugal force (see eq. (7-3)) , the general Bernoulli function (16-30) becomes in the rotating frame,

$$H = \frac{1}{2}\boldsymbol{u}^2 + w(p) - \frac{1}{2}\Omega^2 r^2 \ , \tag{17-55}$$

where $w(p)$ is the pressure function (16-31), which for incompressible fluids is simply $w = p/\rho_0$. The constancy of H along a streamline in nearly ideal steady flow, creates a relation between inlet and outlet pressures. Bernoulli's theorem should should, however, be used with some caution and will not be fulfilled if a significant amount of energy is dissipated into heat through turbulence.

Example 17.8.1 (Lawn sprinkler, continued): Returning to the lawn sprinkler in example 17.7.1 we now assume that the water feeds into sprinkler pivot through a half inch tube. Ignoring the square of the small inlet velocity (about 2 m/s), Bernoulli's theorem implies that in the steady state the inlet pressure excess is

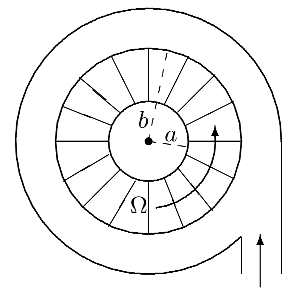
$$\Delta p = p - p_0 \approx \frac{1}{2}\rho_0 \left(U^2 - \Omega_0^2 R^2\right) \approx 0.5 \text{ bar} \ . \tag{17-56}$$

The total power consumed by the sprinkler is determined by the pressure excess (and thus paid for by the waterworks). It is $P_0 = \Delta p \, Q/\rho_0 \approx 12$ W and mostly goes to provide the kinetic energy of the sprinkled water. The total shaft power (lost to friction) becomes $P = -N\Omega_0 \approx -1.3$ W which is only a tenth of the total power consumption.

Radial flow rotor

Radial flow rotors are mostly used in high-pressure turbomachinery, such as pumps, compressors, and hydraulic turbines. Although the following analysis concerns a turbine it is also valid for a pump. We assume that the flow is effectively incompressible and nearly ideal.

Let the rotor have inner radius a, outer radius b, and axial length L. We shall for simplicity assume that it is perfectly cylindrical with the same properties along the whole of its length $0 < z < L$, so that the channels have the same shape for all z. In a turbine, high-speed water is fed in at the outer perimeter $r = b$ with (negative) radial velocity $u_r(b) = v_r(b)$ and removed at the inner perimeter $r = a$ with (negative) radial velocity $u_r(a) = v_r(a)$. Disregarding the blocking from the thin blades separating the otherwise identical channels, the total mass flow is approximatively,



Sketch of a radial flow turbine rotating steadily with angular velocity Ω. The z-axis comes out of the paper. Fluid enters the rotor at r = b, moves radially inwards, and leaves at r = a where it is diverted away from the system along the axis.

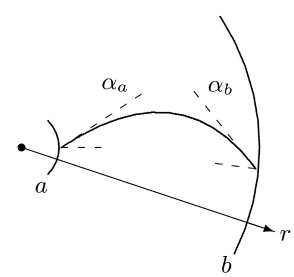
$$Q = 2\pi a L \rho_0 u_r(a) = 2\pi b L \rho_0 u_r(b) \; . \tag{17-57}$$

The integral in the shaft moment (17-53) only receives contributions from the radial surfaces at $r = a, b$. Since v_ϕ is approximatively independent of ϕ and z, we find the shaft power (*Euler's turbine equation*),

$$P = (b\, v_\phi(b) - a\, v_\phi(a))\Omega Q \; . \tag{17-58}$$

For a turbine where Q is negative, the shaft power will be negative as long as the angular momentum density is larger at the entry than at the exit, $b\, v_\phi(b) > a\, v_\phi(a)$. The fluid understandably has to loose angular momentum for the rotor to produce the work that may drive an electric generator.

The rotor consists of a number of thin blades mounted radially on a hub like the spokes of a wheel. Let the rotor blades be designed such that the geometric slopes with respect to the radial direction are α_a at $r = a$ and α_b at $r = b$. In smooth operation we assume that the steady flow enters and leaves tangentially along the blades. Although a rotor can still operate if this condition is not fulfilled, it is much more liable to generate turbulence, accompanied by loss of power, because of the sudden change in flow direction, especially at the entry. The smoothness condition provides the following relations between the relative radial and azimuthal velocities,



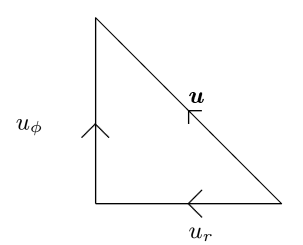
Sketch of a curved blade with slope α_a at $r = a$ and α_b at $r = b$ relative to the radial direction. Here α_a is positive and α_b negative.

$$u_\phi(a) = \alpha_a u_r(a) \; , \qquad\qquad u_\phi(b) = \alpha_b u_r(b) \; . \tag{17-59}$$

Using that the absolute azimuthal velocity is $v_\phi(r) = u_\phi(r) + \Omega r$, and eliminating the azimuthal velocities by means of these equations and the radial velocities by means of (17-57), we obtain

$$\boxed{P = \left((b^2 - a^2)\Omega + \frac{Q}{2\pi L \rho_0}(\alpha_b - \alpha_a)\right)\Omega Q \; .} \tag{17-60}$$

Even if the blades have no curvature, $\alpha_a = \alpha_b$, the shaft power will be non-vanishing. If $\alpha_b < \alpha_a$, both terms in the parenthesis are positive (for a turbine).



The relation between radial velocity $u_r = v_r$ and azimuthal velocity $u_\phi = v_\phi - \Omega r$ in the rest system of the rotor. The slope is defined to be $\alpha = u_\phi/u_r$.

Applying Bernoulli's theorem in the rotating frame to a streamline passing through a channel, it follows from (17-55) that

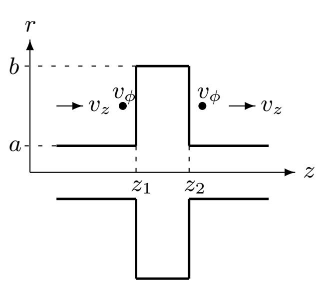
$$\frac{1}{2}\boldsymbol{u}(a)^2 + \frac{p(a)}{\rho_0} - \frac{1}{2}\Omega^2 a^2 = \frac{1}{2}\boldsymbol{u}(b)^2 + \frac{p(b)}{\rho_0} - \frac{1}{2}\Omega^2 b^2 \ .$$

Using that $\boldsymbol{u}^2 = u_r^2 + u_\phi^2$ we obtain after elimination of the velocities

$$p(b) - p(a) = \frac{1}{2}\rho_0 \left(\Omega^2 \left(b^2 - a^2\right) + \frac{Q^2}{(2\pi L \rho_0)^2} \left(\frac{1+\alpha_a^2}{a^2} - \frac{1+\alpha_b^2}{b^2} \right) \right) \ . \quad (17\text{-}61)$$

The first term is due to the centrifugal force and the second to the change in velocity through the rotor channel. If the slopes are of nearly equal magnitudes, $|\alpha_a| \approx |\alpha_b|$ both terms will be positive so that the pressure will always be largest at the outer rim of the rotor. This is the pressure head which drives a hydraulic turbine powered by high-pressure water coming down from a reservoir. Conversely, this is also the pressure head created by a pump used to lift water up into a reservoir.

Example 17.8.2 (Hydraulic turbine): A large hydraulic turbine is driven by water piped down from a reservoir $h = 110$ m above the turbine. The rotor axis is vertical and the inner and outer radii are $a = 2$ m and $b = 3$ m. The axial length is $L = 0.5$ m, and the rotor rotates at 60 rpm, *i.e.* once per second or $\Omega = 2\pi$ s^{-1}. The blades are constructed with vanishing entry slope $\alpha_b = 0$, so that the azimuthal inlet velocity is $v_\phi(b) = \Omega b \approx 19$ m/s. At the exit the slope is chosen to be $\alpha_a = 1/3$, and the flow is required to be purely radial, $v_\phi(a) = 0$, leading to $u_r(a) = -\Omega a/\alpha_a \approx -38$ m/s and $u_r(b) = a u_r(a)/b \approx -25$ m/s. The absolute inlet velocity of the water becomes $v(b) \approx 31$ m/s, corresponding to a free fall from 50 m height. The flux of water may now be calculated from (17-57) and becomes $Q/\rho_0 \approx -237$ m^3/s. The total shaft power (17-60) is $P \approx -84$ MW. The pressure drop through the turbine is about $\Delta p = p(b) - p(a) \approx 5.7$ bar which is a bit more than half the static pressure head from the reservoir. The total rate of work of the excess pressure is $P_0 = \Delta p \, Q/\rho_0 \approx 136$ MW, so that the shaft work of the turbine is about $P/P_0 \approx 62\%$ of the total work of the water. The remainder is found in the kinetic energy of the water coming out of the turbine at higher speed than it entered. In this calculation we have disregarded losses due to friction and turbulence which can be quite significant.



Sketch of axial flow rotor. The control volume is defined by $z_1 < z < z_2$ and $a < r < b$. The axial flow $v_z(r)$ is independent of z, whereas the azimuthal flow $v_\phi(r,z)$ may depend on both r and z.

Axial flow rotor

Axial flow rotors are typically used in high-volume turbomachinery, such as fans and low-pressure turbines. The fluid is also in this case taken to be nearly ideal and effectively incompressible with constant density, $\rho = \rho_0$. Since there is no radial flow in this design, mass conservation guarantees that the axial velocity $v_z = v_z(r)$ will be independent of z. The rotor is assumed to be cylindrical with inner and outer radii a and b, and thin blades that do not significantly obstruct the flow. The fluid enters the rotor at $z = z_1$ and leaves at $z = z_2$. We shall for simplicity assume that the axial flow is uniform $v_z = U$, so that the total mass flow through the rotor becomes

$$Q = \int_a^b \rho_0 v_z(r)\, 2\pi r\, dr = \rho_0 U A \ , \tag{17-62}$$

where $A = \pi(b^2 - a^2)$ is the area of the axial rotor cross section.

The rotor channels change the azimuthal velocity field $v_\phi(r, z)$. The integral in the shaft moment (17-53) only receives contributions from the inlet at $z = z_1$ and the outlet at $z = z_2$, so that the shaft power becomes

$$P = \int_a^b \rho_0 r \big(v_\phi(r, z_2) - v_\phi(r, z_1)\big) v_z(r)\, 2\pi r\, dr = \langle r(v_{\phi 2} - v_{\phi 1})\rangle\, \Omega Q \ . \tag{17-63}$$

In the last step we have expressed the integral in terms of the area average $\langle f \rangle = \frac{1}{A}\int_a^b f(r) 2\pi r\, dr$ over the cross section of the rotor. The shaft power is positive if the azimuthal speed generally increases through the rotor, as one would also expect.

In steady operation we assume that the flow enters and leaves the spinning rotor tangentially along the blades. The rotor blades are designed such that the slope with respect to the z-axis is $\alpha_1(r)$ at the entrance and $\alpha_2(r)$ at the exit. Since the relative azimuthal rotor speed is $u_\phi = v_\phi - \Omega\, r$, the smoothness condition amounts to,

$$u_{\phi 1} = v_{\phi 1} - \Omega r = \alpha_1 U \ , \qquad u_{\phi 2} = v_{\phi 2} - \Omega r = \alpha_2 U \ . \tag{17-64}$$

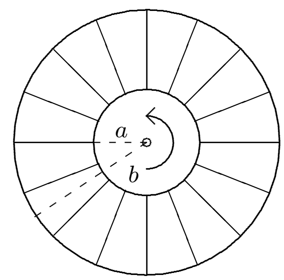
Using these relations to eliminate the azimuthal velocities, the shaft power may be written

$$\boxed{P = \langle r(\alpha_2 - \alpha_1)\rangle\, U \Omega Q \ .} \tag{17-65}$$

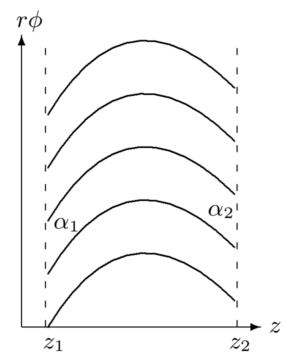
The slope average $\langle r(\alpha_2 - \alpha_1)\rangle$ is a purely geometric factor which is independent of both U and Ω. Notice that this expression is only valid if the smoothness condition is actually being fulfilled.

If the inlet and exit slopes are the same $\alpha_1 = \alpha_2$, the total shaft power appears to vanish. This seems a bit puzzling, because it is well-known that propellers with flat blades also require engine power to turn steadily. The explanation is that for such devices the smoothness condition is not fulfilled, and the fluid hits the propeller blades at an "angle of attack", resulting in aerodynamic forces which oppose the motion and thus require power in the steady state (aerodynamic forces are discussed at length in chapter 27).

The actual geometric design of the blades provides us with the slopes. Here we shall for simplicity only consider the case, in which the blades have slopes that grow proportionally with the radial distance r, *i.e.* $\alpha_1 = \beta_1 r$ and $\alpha_2 = \beta_2 r$. Under this assumption, the fluid will rotate like a solid body with $v_{\phi 1} = \Omega_1 r$



Rotor design with blades arranged like the spokes of a wheel. The positive z-axis comes out of the paper, and the indicated sense of rotation is positive.



The axial flow rotor seen edgewise, folded out along a constant radius r with the r-axis going into the paper. The blades have slope α_1 relative to the z-axis at the entrance $z = z_1$ and α_2 at the exit $z = z_2$; here shown with $\alpha_1 > 0$ and $\alpha_2 < 0$.

at the entrance and $v_{\phi 2} = \Omega_2 r$ at the exit, where the smoothness condition determines the angular velocities,

$$\Omega_1 = \Omega + \beta_1 U \ , \qquad \Omega_2 = \Omega + \beta_2 U \ . \tag{17-66}$$

Now the shaft power (17-65) can be calculated explicitly with the result

$$P = \frac{1}{2}(a^2 + b^2)(\beta_2 - \beta_1)U\Omega Q \ . \tag{17-67}$$

If the blades are not designed with uniformly growing slopes, this expression may still be used for the purpose of doing estimates.

Finally, the pressure change through the rotor is obtained from the Bernoulli function (17-55) in the rest system of the rotor. For a streamline at constant r passing through an axial channel from inlet to outlet we find

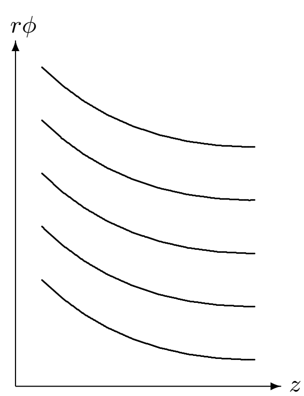
$$\frac{1}{2}\boldsymbol{u}_1^2 + \frac{p_1}{\rho_0} - \frac{1}{2}\Omega^2 r^2 = \frac{1}{2}\boldsymbol{u}_2^2 + \frac{p_2}{\rho_0} - \frac{1}{2}\Omega^2 r^2 \ .$$

Using that $\boldsymbol{u}^2 = u_\phi^2 + u_z^2$ and $u_z = v_z = U$, we arrive at

$$p_1 - p_2 = \frac{1}{2}\rho_0(\alpha_2^2 - \alpha_1^2)U^2 = \frac{1}{2}\rho_0(\beta_2^2 - \beta_1^2)r^2U^2 \ . \tag{17-68}$$

The pressure difference between inlet and outlet grows in this case with the square of the distance of the channel from the axis.

Axial flow turbines are often provided with upstream *guide vanes* to add an initial azimuthal spin Ω_1 to the fluid before it enters the rotor. The guide vanes have basically the same function as the rotor blades, but since they are solidly anchored to the housing of the machine the reaction moment from the fluid on the guide vanes plays no role for the shaft power. For axial flow blowers downstream guide vanes may similarly be used "straighten out" the rotating fluid before it exits from the blower.



The blades in the air blower of example 17.8.3 folded out along the rotor circumference. The blades move upwards here.

Example 17.8.3 (Air blower): In longer road tunnels axial flow air blowers are often used to create an artificial draught along the tunnel in order to rid it of exhaust fumes, especially when traffic has stalled. Suppose an air blower with outer diameter $2b = 60$ cm and hub diameter $2a \approx 20$ cm operates at $\Omega = 600$ rpm $= 10 \cdot 2\pi$ s^{-1}. The blades are constructed with radially increasing slopes, and the maximal entrance angle at the tip of a blade is $-45°$, corresponding to $\alpha_1(b) = -1$ and thus $\beta_1 = \alpha_1(b)/b \approx -3.33$ m^{-1}. At the entrance the air has no spin, $\Omega_1 = 0$, and the axial flow becomes $U = -\Omega/\beta_1 \approx 19$ m/s. At the exit, the blade slope vanishes, $\beta_2 = 0$, so that $\Omega_2 = \Omega$. The total volume flow becomes $Q/\rho_0 \approx 4.7$ m3/s and the total shaft power becomes $P \approx 1$ kW. The maximal pressure increase across the blower is found at the tip of the blades, $r = b$, and is merely $\Delta p = p_2 - p_1 \approx 195$ Pa ≈ 2 mbar.

* Rotor scaling relations

Although it is possible to estimate the overall behavior of a rotor from angular momentum balance and Bernoulli's theorem, the details of the flow pattern through the actual rotor and the unavoidable viscosity of the fluid will give rise to important corrections to the shaft power and the pressure jump across the rotor, corrections that may not be easily calculable from theory. Model studies of scaled down versions of the rotor may, however, be feasible and the measured values of shaft power and pressure jump can afterwards be scaled up to yield a prediction for the actual machinery. So we need to know the scaling relations for these quantities.

A rotor of a given design is characterized by a single length scale D, the "diameter" of the system. The fluid itself is characterized by its density ρ in some point, say at the inlet. We shall furthermore assume that the viscosity is so small that it can be disregarded. During steady operation of the rotor, the controlling parameters are the angular velocity Ω and mass flux Q. Since D controls the length scale, Ω the time scale, and ρ the mass scale, the only dimensionless parameter which can be constructed is,

$$q = \frac{Q}{\rho \Omega D^3} \ . \tag{17-69}$$

It may be viewed as a dimensionless representation of the mass flux and is usually called the *flow coefficient*. It also represents the ratio $q \sim u/v$ between the relative flow velocity $u \sim Q/\rho D^2$ and the absolute velocity $v \sim \Omega D$.

From (17-53) and (17-54) it follows that the shaft power scales like $P \sim \Omega D v Q \sim \Omega^2 D^2 Q$. Similarly, it follows from (17-55) that the pressure head scales like $\Delta p \sim \frac{1}{2}\rho\Omega^2 D^2$. The exact equations of fluid mechanics (without viscosity) must therefore provide relationships of the form

$$\frac{P}{\Omega^2 D^2 Q} = f(q) \ , \qquad \frac{\Delta p}{\frac{1}{2}\rho \Omega^2 D^2} = g(q) \ , \tag{17-70}$$

where $f(q)$ and $g(q)$ are — generally unknown — dimensionless functions of the dimensionless control variable q. For a radial rotor with nearly ideal incompressible flow, we take $D = b$ and find from (17-60) and (17-61),

$$f = 1 - \frac{a^2}{b^2} + q\frac{b}{2\pi L}(\alpha_b - \alpha_a) \ , \quad g = 1 - \frac{a^2}{b^2} + q^2 \frac{b^4}{(2\pi L)^2}\left(\frac{1+\alpha_a^2}{a^2} - \frac{1+\alpha_b^2}{b^2}\right) \ .$$

If viscosity is important, f and g will also depend on the Reynolds number.

Example 17.8.4 (Model turbine): For the hydraulic turbine of example 17.8.2 we take $D = b$ and find $q = 1.4$. A model turbine scaled down by a factor 10, *i.e.* with $D \to 0.1D$, but operated at the same angular velocity Ω and the same value of q has a thousand times smaller mass flow $Q \to 10^{-3}\,Q$. The scaling relations now imply that $P \to 10^{-5}P$ and $\Delta p \to 10^{-2}\Delta p$, independently of the form of the generally unknown functions f and g. Measuring the values of P and Δp in the model turbine immediately allows us to infer the values for the full-scale turbine (the Reynolds number may, however, be different for turbine and model).

17.9 Kinetic energy balance

The kinetic energy of a material particle with mass dM is

$$d\mathcal{T} = \frac{1}{2}\boldsymbol{v}^2\,dM = \frac{1}{2}\rho\boldsymbol{v}^2\,dV\ , \tag{17-71}$$

and the total kinetic energy of all the material particles in a control volume becomes

$$\mathcal{T} = \int_V \frac{1}{2}\rho\boldsymbol{v}^2\,dV\ . \tag{17-72}$$

It must be emphasized that this is only the kinetic energy of the mean flow of matter represented by the velocity field $\boldsymbol{v}$. There is definitely also kinetic energy associated with the fast random motion of the molecules relative to the mean flow (see problem 17.10). This "hidden" contribution to the kinetic energy contained in any piece of matter is known to us as *heat*. The proper discussion of heat in the context of the First Law of Thermodynamics will be postponed to chapter 28.

Global equation of kinetic energy balance

The material derivative of the kinetic energy is calculated in the same way as for momentum (17-13) with the result,

$$\frac{D\mathcal{T}}{Dt} = \int_V \rho\frac{D(\frac{1}{2}\boldsymbol{v}^2)}{Dt}\,dV \tag{17-73}$$

Carrying out the derivative we find by means of the fundamental dynamical equation of continuous matter (15-35)

$$\rho\frac{D\left(\frac{1}{2}\boldsymbol{v}^2\right)}{Dt} = \rho\boldsymbol{v}\cdot\frac{D\boldsymbol{v}}{Dt} = \boldsymbol{v}\cdot\boldsymbol{f}^*\ . \tag{17-74}$$

Thus, the *global equation of kinetic energy balance* takes the form

$$\boxed{\frac{D\mathcal{T}}{Dt} = P}\ , \tag{17-75}$$

where

$$P = \int_V \boldsymbol{v}\cdot\boldsymbol{f}^*\,dV \tag{17-76}$$

is the total rate of work, or *power*, of all the forces acting on the material particles in the control volume[1].

[1]There is a slight clash between the symbol for total momentum $\mathcal{P}$ and for total power P, but it is resolved by consistently denoting momentum by the script version of the letter.

Reynolds theorem for kinetic energy is obtained from (17-9) by replacing ρ with $\frac{1}{2}\rho\,\boldsymbol{v}^2$,

$$\boxed{\frac{D\mathcal{T}}{Dt} = \frac{d\mathcal{T}}{dt} + \oint_S \frac{1}{2}\rho\,\boldsymbol{v}^2\,(\boldsymbol{v} - \boldsymbol{v}_S)\cdot d\boldsymbol{S}\ .} \tag{17-77}$$

To derive a global expression for the power we use the expression (9-18) for the effective force density to write,

$$\boldsymbol{v}\cdot\boldsymbol{f}^* - \boldsymbol{v}\cdot\boldsymbol{f} = \sum_{ij} v_i \nabla_j \sigma_{ij} = \sum_{ij} \nabla_j (v_i \sigma_{ij}) - \sum_{ij} \sigma_{ij} \nabla_j v_i\ .$$

The integral of the first term in the final result may be converted to a surface integral by means of Gauss' theorem (6-4) to obtain,

$$\boxed{P = \int_V \boldsymbol{v}\cdot\boldsymbol{f}\,dV + \oint_S \boldsymbol{v}\cdot\boldsymbol{\sigma}\cdot d\boldsymbol{S} - \int_V \sum_{ij} \sigma_{ij} \nabla_j v_i\,dV\ .} \tag{17-78}$$

The total power thus consists of the power of the volume forces (external as well as internal), the power of the external contact forces, plus a third contribution

$$P_{\text{int}} = -\int_V \sum_{ij} \sigma_{ij} \nabla_j v_i\,dV\ , \tag{17-79}$$

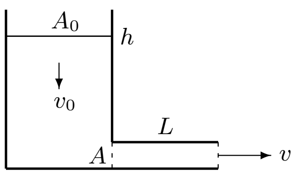
which we interpret as the *power of the internal contact forces*. Although Newton's third law guarantees that the contact forces between neighboring material particles cancel, the power of these forces may not cancel because neighboring material particles will have slightly different velocities, as witnessed by the appearance of the velocity gradients $\nabla_j v_i$ in the last term.

We shall later see that the rate of work of internal contact forces has essentially only two contributions: the work performed under local deformation of the material and the work done by local friction forces. Kinetic energy lost to deformation is usually stored as elastic internal energy and may be converted back to kinetic energy a short time later. This is for example what happens in an isentropic pressure wave in an ideal fluid where there is a rapid alternation between motion and pressure (see section 16.2). Since local compression heats the fluid, heat conduction may however prevent the full recovery of the local kinetic energy over longer times. Kinetic energy lost to local friction will on the other hand turn directly into heat which in general cannot be fully recovered as kinetic energy. Explicit expressions for the internal power in viscous fluids will be given in chapter 18 and the dynamics of heat will be discussed in chapter 28.

Total power in an ideal fluid

In an ideal (Euler) fluid all stresses are due to pressure, $\sigma_{ij} = -p\,\delta_{ij}$, and the total power (17-78) takes the simpler form,

$$P = \int_V \boldsymbol{v} \cdot \boldsymbol{f}\, dV - \oint_S p\, \boldsymbol{v} \cdot d\boldsymbol{S} + \int_V p\, \boldsymbol{\nabla} \cdot \boldsymbol{v}\, dV \ . \tag{17-80}$$



The water accelerates in the pipe while pouring out through the exit.

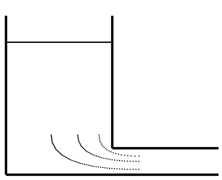
Since the local rate of volume change at the surface of a comoving control volume is $\boldsymbol{v} \cdot d\boldsymbol{S}$, the second term can be identified with the thermodynamic rate of work $(-p\,dV/dt)$ of the external surface pressure distribution. The last term is only non-zero for compressible fluids, and yields a positive contribution when the material locally expands ($\boldsymbol{\nabla} \cdot \boldsymbol{v} > 0$) under positive internal pressure ($p > 0$).

Pulling the plug

When the plug is pulled in the bottom of a cistern filled with water, the fluid goes through a short initial acceleration before steady flow is established. For simplicity we assume that the drain is a horizontal pipe of length L and cross section A, that the cistern is vertical with large cross section $A_0 \gg A$, and that the cistern is continuously being refilled to maintain a constant water level h. Initially the pipe is full and all the water is assumed to be at rest. The control volume is fixed and contains all the water in the system between the open surface, the pipe outlet, and the walls of the cistern and pipe.

From mass conservation we know that the volume of incompressible water which at any moment runs out of the pipe equals the volume put back into the cistern to keep the level h. If v is the (average) horizontal velocity in the pipe and v_0 the (average) vertical velocity in the cistern, Leonardo's law (page 241) tells us that $Av = A_0 v_0$. Assuming that the water is also nearly ideal, Torricelli's law (page 266) tells us that the steady terminal speed, $\sqrt{2g_0 h}$, is the same as the speed the water would obtain by falling freely from the top of the water surface in the cistern to the exit of the pipe. We shall now calculate how the water arrives at this speed.

The total kinetic energy of the water in the system is the sum of the kinetic energies of the water moving vertically through the cistern and the water moving horizontally through the pipe,



The kinetic energy of the water in the transition region between cistern and pipe is assumed to be negligible.

$$\mathcal{T} \approx \frac{1}{2}\rho_0 A_0 h\, v_0^2 + \frac{1}{2}\rho_0 A L v^2 \approx \frac{1}{2}\rho_0 A L v^2 \left(1 + \frac{hA}{LA_0}\right) \approx \frac{1}{2}\rho_0 A L v^2 \ . \tag{17-81}$$

In the last step we have assumed that $hA \ll LA_0$, *i.e.* that water in the cistern moves so slowly that its kinetic energy is negligible (in spite of its larger volume) compared to that of the water in the pipe .

In deriving this equation we have made the usual approximation of equating the averages of products with the products of averages. Since we do not know the actual flow pattern in the cistern near the drain without solving Euler's equation, we shall assume that the transition from v_0 in the cistern to v in the drain happens in a small volume of dimensions comparable to the linear size of the drain, say $\sqrt{A}$. Provided $L \gg \sqrt{A}$, it should be permissible to ignore the kinetic energy of the water in the transition region in comparison with the kinetic energy of the water in the pipe (see problem 17.7).

Kinetic energy is lost through the outlet A and gained through the inlet A_0, so that the net rate of loss of kinetic energy from the system is (under the same approximations as above),

$$\oint_S \frac{1}{2}\rho_0 \boldsymbol{v}^2 \boldsymbol{v} \cdot d\boldsymbol{S} \approx \frac{1}{2}\rho_0 A v^3 - \frac{1}{2}\rho_0 A_0 v_0^3 = \frac{1}{2}\rho_0 A v^3 \left(1 - \frac{A^2}{A_0^2}\right) \approx \frac{1}{2}\rho_0 A v^3 \ .$$

Consequently, the material rate of change of the kinetic energy (17-77) becomes

$$\frac{D\mathcal{T}}{Dt} = \rho_0 A L v \frac{dv}{dt} + \frac{1}{2}\rho_0 A v^3 \ . \tag{17-82}$$

Gravity performs work at the rate $\rho_0 A_0 h g_0 v_0$ on the water moving vertically downwards in the cistern. At the open surface and at the pipe outlet, there is atmospheric pressure p_0 which performs positive work $A_0 p_0 v_0$ on the water coming in and negative work $-A p_0 v$ at the water going out. Since the water is assumed to be incompressible ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) and nearly ideal, there is no kinetic energy lost to compression or heat, and the total power (17-80) of all the forces becomes,

$$P \approx \rho_0 A_0 h g_0 v_0 + A_0 p_0 v_0 - A p_0 v = \rho_0 A h g_0 v \ . \tag{17-83}$$

In the last step we have again used Leonardo's law.

Putting it all together, kinetic energy balance results in the differential equation for the drain pipe velocity,

$$\frac{dv}{dt} = \frac{2g_0 h - v^2}{2L} \ . \tag{17-84}$$

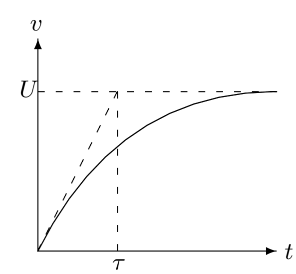
The right hand side vanishes for $v = \sqrt{2g_0 h}$, which is the terminal velocity. Solving the differential equation with the initial condition $v = 0$ at $t = 0$, we find

$$\boxed{v(t) = \sqrt{2g_0 h} \tanh \frac{t}{\tau} \ ,} \tag{17-85}$$

where

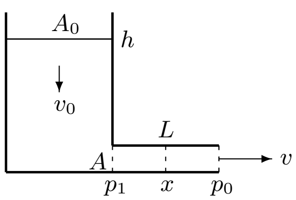
$$\tau = \frac{2L}{\sqrt{2g_0 h}} \ , \tag{17-86}$$

is the characteristic rise time towards terminal velocity. Quite reasonably, it equals the time it takes the water to pass through the pipe at half the terminal speed. At $t = 2\tau$, the water has reached 96 % of terminal speed.



Rise of the velocity in the drain pipe towards terminal velocity $U = \sqrt{2g_0 h}$.

Example 17.9.1 (Barrel of wine): For a cylindrical barrel of wine (example 16.3.2) with diameter 1 m and height 2 m, emptied through a spout with diameter 5 cm and length $L = 20$ cm, the terminal speed is 6 m/s, and the water obtains 96 % of this speed speed after just 0.13 seconds! If you try to put back the plug in the spout after merely half a second, there will nevertheless arrive about 5 liters of wine on the floor (and on you).



During acceleration there is a pressure drop $\Delta p = p_1 - p_0$ between the entry and exit to the pipe.

Until now we have completely escaped the problem of how the pressure behaves inside the cistern and in the pipe. Taking the x-axis along the pipe, and using that v does not depend on x, there will be no comoving acceleration in the pipe, and the x-component of Euler's equation (16-1) becomes simply

$$\frac{dv}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{17-87}$$

Since dv/dt takes the same value everywhere along the pipe, the pressure gradient will be independent of x. Expressed in terms of the pressures p_1 and p_0 at the ends of the pipe, it is $\partial p/\partial x = (p_0 - p_1)/L$. Finally, inserting the acceleration (17-84) and the solution (17-85) we obtain the pressure difference between the pipe entry and the exit

$$\Delta p = p_1 - p_0 = \frac{1}{2}\rho_0(2g_0 h - v^2) = \frac{\rho_0 g_0 h}{\cosh^2(t/\tau)} \ . \tag{17-88}$$

At $t = 0$ the entry pressure p_1 equals the hydrostatic pressure of the water in the cistern, as one would expect, and it approaches atmospheric pressure p_0 everywhere in the pipe in the characteristic rise time τ. In reality there will remain a residual pressure drop in the pipe due to viscous friction (section 19.6).

17.10 Mechanical energy balance

Let us for simplicity assume that the volume forces are entirely due to an external *static* field of gravity, $\boldsymbol{f} = \rho\boldsymbol{g}$ with $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi(\boldsymbol{x})$. In such a field a material particle has potential energy $\Phi\, dM$ (see section 3.5) and the *total potential energy* of the material in the control volume becomes

$$\mathcal{V} = \int_V \rho\,\Phi\, dV \ . \tag{17-89}$$

Using the by now familiar method (leading to (17-13)), the material time derivative of the potential energy becomes

$$\frac{D\mathcal{V}}{Dt} = \int_V \rho \frac{D\Phi}{Dt}\, dV = \int_V \rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\Phi\, dV = -\int_V \rho\,\boldsymbol{v}\cdot\boldsymbol{g}\, dV \ .$$

Since the final result is the opposite of the volume force contribution to (17-80), it makes sense to define the *total mechanical energy* as the sum of the kinetic and potential energies,

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \int_V \rho \left(\frac{1}{2}\boldsymbol{v}^2 + \Phi\right) dV \ . \tag{17-90}$$

Mechanical energy balance now follows from kinetic energy balance (17-75)

$$\boxed{\frac{D\mathcal{E}}{Dt} = \widetilde{P} \ ,} \tag{17-91}$$

where the right hand side is the power (17-78) of all forces, excluding gravity.

For an ideal fluid, this *reduced power* becomes

$$\widetilde{P} = -\oint_S p\,\boldsymbol{v} \cdot d\boldsymbol{S} + \int_V p\,\boldsymbol{\nabla} \cdot \boldsymbol{v}\, dV \ . \tag{17-92}$$

The reduced power will vanish if the pressure does no net work on the surface, and if the fluid is incompressible. When the reduced power vanishes, the total mechanical energy in a comoving control volume will be constant, and this was the principle we used intuitively in the analysis of connected tubes in the beginning of this chapter (section 17.1).

Pulling the plug, again

It is instructive to see how this works out for the case of pulling the plug discussed in the previous section, where the gravitational potential energy may be calculated from the total mass ($\rho_0 A_0 h$) of the water in the barrel times the height ($\frac{1}{2}h$) of the center of mass,

$$\mathcal{V} \approx \frac{1}{2}\rho_0 A_0 g_0 h^2 \ . \tag{17-93}$$

The potential energy of the water in the pipe does not contribute because the baseline for the gravitational potential has been chosen to be $z = 0$. This also guarantees that there is no loss of gravitational potential energy through the drain, such that

$$\frac{D\mathcal{V}}{Dt} = \frac{d\mathcal{V}}{dt} = \rho_0 A_0 g_0 h \frac{dh}{dt} = -\rho_0 A_0 g_0 h v_0 \ .$$

Since this is exactly the opposite of the gravitational term in the total power (17-83), the two methods lead (of course) to the same dynamical equation (17-84).

The total reduced power vanishes in this case, $\widetilde{P} = 0$, but even if mechanical energy balance (17-91) implies that the material rate of change of the energy vanishes, $\mathcal{D}\mathcal{E}/Dt = 0$, this should not be construed to mean that the total mechanical energy of the system is constant. The control volume is in this case neither fixed nor comoving, and energy is evidently lost through the drain.

17.11 Energy balance in an elastic fluid

The global equation of mechanical energy balance (17-91) arises from kinetic energy balance (17-75) by — so to speak — "moving over" the rate of work of gravity from the right hand side of the equation to the left hand side and expressing it as the material derivative of the potential energy. We shall now see that it is also possible to "move over" the last term in the reduced power (17-92) for inviscid barotropic compressible fluids[2]. This procedure will add an extra term to the total energy, representing the internal compression energy of the fluid. Since this energy is analogous to the potential energy stored in a compressed spring, barotropic fluids are often called *elastic fluids*.

The temperature of a fluid normally rises during compression and falls during expansion. Since heat will always flow from hot to cold, temperature differences between different regions tend to be smoothed out by heat flow. Thus, for a fluid to be truly elastic, we must also require that it does not conduct heat, so that the flow is adiabatic (isentropic). Heat flow will be included in chapter 28.

Internal energy

The requirement of isentropic flow leads to another relation besides the equation of state, so that there is only one free thermodynamic variable: ρ, p, or T. In isentropic flow all fluids may thus be viewed as barotropic, $\rho = \rho(p)$, and we have seen before (page 70 and 270) that the *pressure function*,

$$w(p) = \int \frac{dp}{\rho(p)} \ , \tag{17-94}$$

plays an important role. To see how w relates to the internal energy, we begin by deriving the following relation,

$$\rho \frac{D}{Dt}\left(w - \frac{p}{\rho}\right) = \rho\left(\frac{1}{\rho}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt}\right) = \frac{p}{\rho}\frac{D\rho}{Dt} = -p\boldsymbol{\nabla}\cdot\boldsymbol{v} \ .$$

In the last step we have used the continuity equation (15-27) which expresses the material derivative as $D\rho/Dt = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v}$. Since the final result is identical to the integrand in the last term in (17-92), apart from sign, we are led to define the quantity,

$$\mathcal{U} = \int_V \rho\left(w - \frac{p}{\rho}\right) dV \ , \tag{17-95}$$

called the *internal energy*. The expression in the parenthesis,

$$u = \frac{d\mathcal{U}}{dM} = w - \frac{p}{\rho} \tag{17-96}$$

[2]Incompressible, inviscid fluids are uninteresting in this respect because the last term vanishes, and for viscous fluids the corresponding procedure is much more complicated (see chapter 28).

is called the *specific internal energy*. The fact that it vanishes for incompressible fluids where $w = p/\rho$, confirms that it is related to compression.

For isentropic flow in an ideal gas the pressure function is given by (16-32), and we find the specific internal energy,

$$\boxed{u = \frac{1}{\gamma - 1}\frac{p}{\rho} = c_v T \ , \qquad c_v = \frac{1}{\gamma - 1}\frac{RT}{M_{\mathsf{mol}}} \ ,} \tag{17-97}$$

where γ is the adiabatic index and the constant $c_v = c_p - R/M_{\mathsf{mol}}$ is the specific heat at constant volume (density).

Total energy balance

The *total energy* of the material in the control volume is defined to be the sum of the mechanical and internal energies, and becomes for an elastic fluid,

$$\mathcal{E} = \mathcal{T} + \mathcal{V} + \mathcal{U} = \int_V \rho\,\epsilon\,dV \ . \tag{17-98}$$

where

$$\epsilon = \frac{d\mathcal{E}}{dM} = \frac{1}{2}\boldsymbol{v}^2 + \Phi + u \tag{17-99}$$

is called the *specific energy*. In the same vein, the specific kinetic energy is $d\mathcal{T}/dM = \frac{1}{2}\boldsymbol{v}^2$ and the specific potential energy $d\mathcal{V}/dM = \Phi$.

Collecting the terms, global energy balance now takes the form

$$\boxed{\frac{D\mathcal{E}}{Dt} = \dot{W} \ ,} \tag{17-100}$$

where

$$\boxed{\dot{W} = -\oint_S p\,\boldsymbol{v}\cdot d\boldsymbol{S} \ ,} \tag{17-101}$$

is the rate of work of the pressure on the surface of the control volume. Since $\delta V = \boldsymbol{v}\delta t \cdot d\boldsymbol{S}$ is the (signed) change in a volume of a comoving surface element $d\boldsymbol{S}$ in the time δt, this expression is identical to the well-known thermodynamic work $-p\,\delta V$ performed on the system, integrated over the surface of the comoving control volume.

For completeness we recapitulate Reynolds theorem for energy,

$$\boxed{\frac{D\mathcal{E}}{Dt} = \frac{d\mathcal{E}}{dt} + \oint_S \rho\,\epsilon\,(\boldsymbol{v} - \boldsymbol{v}_S)\cdot d\boldsymbol{S} \ ,} \tag{17-102}$$

obtained from (17-9) by replacing the mass density ρ by the energy density $\rho\epsilon$. Together with (17-100) and (17-101) this constitutes the basic formalism for energy balance in elastic fluids.

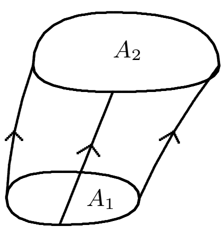
Bernoulli's theorem and energy balance

The specific energy (17-99) may be written

$$\epsilon = H - \frac{p}{\rho} \ , \tag{17-103}$$

where H is the Bernoulli function (16-30) for compressible barotropic fluids,

$$H = \frac{1}{2}\boldsymbol{v}^2 + \Phi + w \ . \tag{17-104}$$



A stream tube consisting of all the flow lines that enter the area A_1 and exit through A_2.

To find the relationship between energy balance and Bernoulli's theorem we assume that the flow is steady and apply energy balance to a fixed control volume in the form of a tiny stream tube consisting of all the streamlines that go into a tiny area A_1 and leave through the equally tiny area A_2. Mass conservation yields

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \ , \tag{17-105}$$

because no fluid passes through the sides of the tube.

The total energy of the fluid in the stream tube must be constant in steady flow, *i.e.* $d\mathcal{E}/dt = 0$, such that the material derivative (17-102) is entirely determined by the rate of loss of energy through the surface of the tube,

$$\frac{D\mathcal{E}}{Dt} \approx \rho_2 \epsilon_2 v_2 A_2 - \rho_1 \epsilon_1 v_1 A_1 \ . \tag{17-106}$$

Similarly, the reduced power (17-101) becomes in the same approximation,

$$\dot{W} = -p_2 v_2 A_2 + p_1 v_1 A_1 \tag{17-107}$$

Energy balance (17-100) implies in conjunction with mass conservation,

$$\epsilon_1 + \frac{p_1}{\rho_1} = \epsilon_2 + \frac{p_2}{\rho_2} \ , \tag{17-108}$$

which in view of the form of the specific energy is nothing but Bernoulli's theorem for a compressible fluid,

$$H_1 = H_2 \ , \tag{17-109}$$

applied to a stream line running through the stream tube.

Bernoulli's theorem is thus completely equivalent to energy balance for steady flow in an ideal non-conducting, *i.e.* elastic fluid. Under these circumstances, we do not gain further information about the system from energy balance than by applying Bernoulli's theorem.

Problems

17.1 Before the advent of modern high precision positioning systems, water levels could be transmitted across a strait by means of a long tube filled with water. Estimate the oscillation time of the water in the tube for the Danish Great Belt which is about 20 km wide.

17.2 Solve the rocket flight problem analytically in the absence of air resistance.

17.3 A horizontal 1 inch water pipe bends horizontally 180°. Estimate the force that the water exerts on the pipe, when the water speed is 1 m/s.

17.4 A firefighter bends a water hose through 90° close to the nozzle where the pressure is nearly atmospheric. The hose diameter is 5 cm and it discharges 40 liter per second. Calculate the magnitude of the force that the firefighter has to exert on the hose to bend it. Will he be able to hold the hose without using equipment?

17.5 Consider a solid rotating with angular velocity Ω around the z-axis. Show that the total power is $P = \Omega\mathcal{M}_z$ where $\mathcal{M}_z$ is the total moment of force along z.

17.6 Use mechanical energy balance (17-91) rather than kinetic energy balance to analyze the initial acceleration of the water in a cistern, when the plug is pulled.

17.7 Consider a circular drain of radius a in the bottom of a large circular cistern of radius $b \gg a$. Assume that the average velocity of the water at a half sphere of radius r centered at the drain is $v(r) = (a/r)^2 v$ where v is the average velocity at the drain, so that the same amount of water passes through the half sphere for all $r > a$. Calculate the kinetic energy associated with this velocity distribution, and compare with the kinetic energies of

17.8 A large lake is drained through a thin pipe that slopes downwards at an angle α with the horizontal. The pipe is positioned near the surface of the lake. A valve is positioned a good distance L down the pipe, but the pipe continues downwards with the same slope far beyond the valve. a) Find the equation of motion for the water front when it has progressed a distance x past the valve. b) Show that the mechanical energy is conserved. c) Use this to calculate the acceleration of the water front as a function of x, and show that terminal speed is never reached.

17.9 A volume force is said to be central, if it acts along the line connecting two material particles and only depends on the distance between them. In that case, the internal density of force in a volume $V(t)$ takes the form

$$\boldsymbol{f}_{\rm int}(\boldsymbol{x},t) = \int_{V(t)} (\boldsymbol{x}-\boldsymbol{y}) h(|\boldsymbol{x}-\boldsymbol{y}|\,,t)\, dV_y \tag{17-110}$$

a) Show that the total internal force vanishes. b) Show that the total internal moment of force also vanishes.

17.10 A volume V contains N point-like particles of mass m_n with instantaneous positions $\boldsymbol{x}_n$ and velocities $\boldsymbol{v}_n$. The total mass is $M = \sum_n m_n$ and the center of mass is $\boldsymbol{x} = \sum_n m_n \boldsymbol{x}_n / M$. Define the relative positions $\boldsymbol{x}'_n = \boldsymbol{x}_n - \boldsymbol{x}$ and the relative velocities $\boldsymbol{v}'_n = \boldsymbol{v}_n - \boldsymbol{v}$ where $\boldsymbol{v} = \sum_n m_n \boldsymbol{v}_n / M$ is the center-of-mass velocity.

Assume that the relative positions and velocities are random and average out to zero,

$$\langle \boldsymbol{x}'_n \rangle = \boldsymbol{0} \tag{17-111}$$

$$\langle \boldsymbol{v}'_n \rangle = \boldsymbol{0} \tag{17-112}$$

Also assume that they are independent, uncorrelated and that the velocities are spherically distributed

$$\langle (\boldsymbol{x}'_n)_i (\boldsymbol{v}'_m)_j \rangle = 0 \tag{17-113}$$

$$\langle (\boldsymbol{v}'_n)_i (\boldsymbol{v}'_m)_j \rangle = U^2 \delta_{ij} \delta_{nm} \tag{17-114}$$

(a) a Show that the total angular momentum of the system is

$$\boldsymbol{\mathcal{L}} = M \boldsymbol{x} \times \boldsymbol{v} + \sum_n m_n \boldsymbol{x}'_n \times \boldsymbol{v}'_n \tag{17-115}$$

and that the second term averages to zero.
(b) bShow that the total kinetic energy is

$$\mathcal{T} = \frac{1}{2} M \boldsymbol{v}^2 + \frac{1}{2} \sum_n m_n (\boldsymbol{v}'_n)^2 \tag{17-116}$$

and calculate the average of the second term.

