14 Elastic vibrations

Sound is the generic term for harmonic running pressure waves in matter, be it solid, liquid or gaseous. Our daily existence as humans, communicating in and out of sight, is strongly dependent on sound transmission in air, and only rarely — as for example in the dentists chair — do we notice primary effects of sound in solids. What we do experience in our daily lives is mostly secondary effects of vibrations in solids transferred to air as sound waves, for example a mouse scratching on the other side of a wooden wall, or more insidiously the neighbor's drilling into concrete. There are also wave motions in elastic solids, for example caused by earthquakes that we would hardly call sound, except sometimes one speaks about infrasound. We don't hear these phenomena directly but rather experience an earthquake as a motion of the ground, though usually accompanied by audible sound.

There are actually two kinds of vibrations in isotropic elastic solids: *longitudinal* pressure waves and *transverse* shear waves. The two kinds of waves are generally transmitted with different phase velocities, because elastic solids respond differently to pressure and shear stress. Vibrations in ideal elastic materials do not dissipate energy, but energy can be lost to spatial infinity through radiation of sound. A church bell or tuning fork may ring for a long time but eventually stops because of radiative and dissipative losses.

In this chapter we shall use Newton's Second Law to derive the basic equations for vibrations in isotropic elastic materials and then apply them to a few generic situations. Elastic vibrations constitute a huge subfield of continuum physics which cannot be given just treatment in a single chapter. The chapter is, however, important because it is the first time we encounter continuous matter in motion, the main theme for the remainder of this book.

14.1 Elastodynamics

The instantaneous state of a deformable material is described by a time-dependent displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$ which indicates how much a material particle at time t is displaced away from its original position $\boldsymbol{x}$. The field $\boldsymbol{u}(\boldsymbol{x}, t)$ should as before be understood as the displacement away from a chosen reference state which may itself already be highly stressed and deformed. There are, for example, huge static stresses in balance with gravity in the pylons and girders of a bridge, but when the wind acts on the bridge, small-amplitude vibrations may arise around the static state.

In this section we shall first establish the fundamental equation of motion for isotropic elastic matter and then draw some general conclusions about the nature of its solutions. Some of the results to be derived will have much wider application than just for elastic waves.

Navier's equation of motion

The actual position of a displaced particle is $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t)$, and since its original position $\boldsymbol{x}$ is time independent, its actual velocity is $\boldsymbol{v}(\boldsymbol{x}, t) = \partial \boldsymbol{u}(\boldsymbol{x}, t)/\partial t$ and its acceleration $\boldsymbol{w}(\boldsymbol{x}, t) = \partial^2 \boldsymbol{u}(\boldsymbol{x}, t)/\partial t^2$. Newton's Second Law — mass times acceleration equals force — applied to every material particle in the body takes the form, $dM\boldsymbol{w} = \boldsymbol{f}^* dV$. Dividing by dV and re-using the effective force density for an isotropic homogeneous elastic material from the left hand side of the equation of equilibrium (12-2), we arrive at *Navier's equation of motion* (1821),

$$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{f} + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u} \ .} \tag{14-1}$$

eNavierMotion

Here λ, μ, and ρ are as before are assumed to be material parameters that do not depend on space and time. In the case that they depend on the spatial position $\boldsymbol{x}$, as they do in Earth's solid mantle, Navier's equation of motion takes a somewhat different form (see problem 14.1). This equation of motion reduces by construction to Navier's equilibrium equation for a time-independent displacement. As in elastostatics, the displacement field and the stress vector must be continuous across material interfaces.

It must be emphasized that Navier's equation of motion is only valid in the limit of small and smooth displacement fields. If the displacement gradients are large, non-linear terms will first of all appear in the strain tensor (10-44), but there will also arise non-linear terms from the derivatives of the stress tensor in the effective force, as demonstrated by eq. (12-3). In chapter 15 we shall derive the correct equations of motion for continuous matter (in the Euler representation) with all such terms included.

Driving forces, dissipation, and free waves

Time-dependent displacement is often caused by contact forces that — like the wind on the bridge — impose time-dependent stresses on the surface of a body. If you hit a nail with a hammer or stroke the strings of a violin, time-varying displacement fields are also set up in the material. Body forces may likewise drive time-dependent displacements. The Moon's tidal deformation of the rotating Earth is caused by time-dependent gravitational body forces, acting on top of the static gravitational force of Earth itself. Magnetostrictive, electrostrictive, and piezoelectric materials deform under the influence of electromagnetic fields, and are for example used in loudspeakers to set up vibrations that can be transmitted to air as sound.

The omnipresent forces of dissipation — not included in Navier's equation of motion — will in the end make all vibrations die out and turn their energy into heat. Sustained vibrations in an any body can strictly speaking only be maintained by time-dependent external forces continually performing work by interacting with the body. Dissipation is nevertheless so small in most elastic materials that it to a very good approximation can be omitted, as it is in Navier's equation of motion. This argument justifies the study of *free elastic waves* in a body subject only to time-independent external forces. Due to the linearity of Navier's equation of motion, a time-independent body force may be removed by means of a suitable time-independent displacement, such that the general equation of motion for free elastic waves becomes,

$$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u} \ .} \qquad (14\text{-}2)$$

eFreeElasticWaves

Although free waves must at some point in time have been created by time-dependent driving forces, they will in a finite, isolated, perfectly elastic body continue indefinitely after the driving forces cease to act.

The violin paradox: How can stroking a violin string with a horsehair bow at constant speed make the string vibrate at a nearly constant frequency, when we claim that sustained vibration demands time-dependent driving forces?

Although the external force delivered by your arm to the bow is nearly constant for the length of the stroke, the interaction between the bow and the string develops time-dependence because of the finite difference between static and dynamic friction forces (section 9.1). The string sticks to the bow when it starts to move until the restoring elastic force in the string surpasses the static friction force, whereupon the string slips and begins to move with much smaller or even no friction (if it lifts off the bow). Swinging once back and forth the string eventually again matches the speed of the bow and sticks. Since it only sticks for a very short time, the frequency generated in this way is very nearly equal to the natural oscillation frequency of a taught but otherwise free string. This *stick-slip* mechanism underlies many oscillatory phenomena apparently generated by steady driving forces (see for example problem 9.8 on page 152).

Longitudinal and transverse waves

An arbitrary vector field may always be resolved into *longitudinal* and *transverse* components (see problem 14.5),

$$\boldsymbol{u} = \boldsymbol{u}_L + \boldsymbol{u}_T \ , \tag{14-3}$$

eLongitudinalTransverse

where the longitudinal component $\boldsymbol{u}_L$ has no curl, and the transverse component $\boldsymbol{u}_T$ has no divergence,

$$\boldsymbol{\nabla} \times \boldsymbol{u}_L = \boldsymbol{0} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u}_T = 0 \tag{14-4}$$

By the "double-cross" rule (2-67) on page 33 it follows that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}_L) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}_L) - \boldsymbol{\nabla}^2 \boldsymbol{u}_L = \boldsymbol{0}$, or $\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u}_L = \boldsymbol{\nabla}^2 \boldsymbol{u}_L$, so that the wave equation (14-2) specialized to purely longitudinal and transverse free waves becomes,

$$\rho \frac{\partial^2 \boldsymbol{u}_L}{\partial t^2} = (\lambda + 2\mu) \boldsymbol{\nabla}^2 \boldsymbol{u}_L \ , \qquad \rho \frac{\partial^2 \boldsymbol{u}_T}{\partial t^2} = \mu \boldsymbol{\nabla}^2 \boldsymbol{u}_T \ . \tag{14-5}$$

Conversely it may be shown that the longitudinal and transverse components of any mixed field (14-3) must also satisfy these equations (see problem 14.5).

Both of these equations are in the form of the standard wave equation,

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = c^2 \boldsymbol{\nabla}^2 \boldsymbol{u} \ , \tag{14-6}$$

eStandardWave

for non-dispersive waves with phase velocity c. For longitudinal and transverse waves the phase velocities are,

$$\boxed{c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}} \ , \qquad \boxed{c_T = \sqrt{\frac{\mu}{\rho}}} \ . \tag{14-7}$$

eElasticWaveVelocities

Material	c_L km/s	q %
Aluminium	6.4	48
Titanium	6.1	51
Iron	5.9	54
Nickel	5.8	52
Magnesium	5.8	54
Quartz	5.5	63
Wolfram	5.2	55
Copper	4.7	49
Silver	3.7	45
Gold	3.6	33
Lead	2.1	33

Longitudinal sound speed and the ratio of transverse to longitudinal speed, $q = c_T/c_L$, for various isotropic materials. The lightest and hardest materials generally have the largest longitudinal sound speed.

In typical elastic materials the phase velocities are a few kilometers per second which is an order of magnitude greater than the velocity of sound in air, but roughly of the same magnitude as the sound velocity in liquids, such as water.

The ratio between the transversal and longitudinal velocities is a useful dimensionless parameter,

$$q = \frac{c_T}{c_L} = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1 - 2\nu}{2(1 - \nu)}} \ . \tag{14-8}$$

eTLratio

It depends only on Poisson's ratio ν, and is a monotonically decreasing function of ν. Its maximal value $\frac{1}{2}\sqrt{3} \approx 0.87$ is obtained for $\nu = -1$, implying that the transverse velocity is always smaller than the longitudinal one. In practice there are no materials with $\nu < 0$, so the realizable upper limit to the ratio is instead $\frac{1}{2}\sqrt{2} \approx 0.71$. For the typical value $\nu = \frac{1}{3}$ we get $q = 1/2$, and longitudinal waves typically propagate with double the speed of transverse waves.

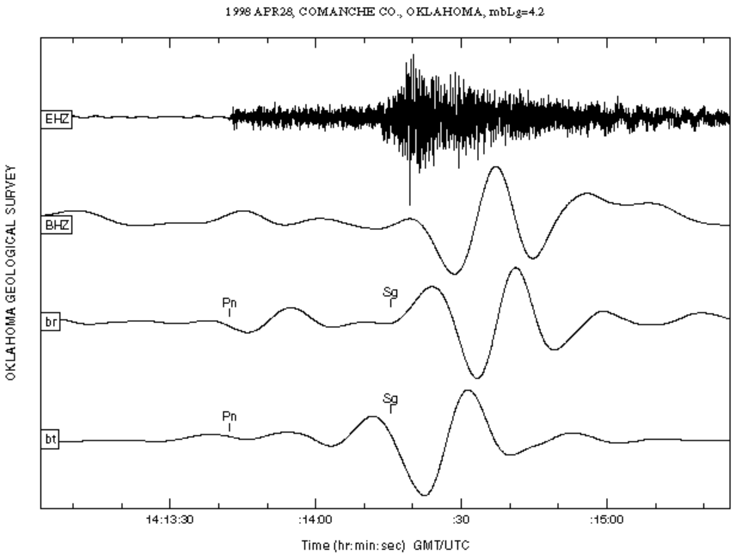


Figure 14.1: *Seismogram of earthquake of size 4.2 that took place in Comanche county, Oklahoma on April 28, 1998. Reproduced here with the permission of the Oklahoma Geological Survey (to be obtained). The four traces are:* ***EHZ*** *vertical earth velocity at all frequencies,* ***BHZ*** *the low-frequency vertical component,* ***br*** *the low-frequency horizontal compressional component (Rayleigh waves), and* ***bt*** *the low-frequency horizontal shear component (Love waves). The times labeled Pn and Sg represent the onset of the primary and secondary disturbances.*

The tiny pressure change (11-14) generated by the displacement field is $\Delta p = -K\boldsymbol{\nabla}\cdot\boldsymbol{u}$, where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus. Since $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ for transverse waves, only the longitudinal waves are accompanied by an oscillating pressure. They are for this reason also called *pressure* waves or *compressional* waves.Transverse waves generate no pressure changes in the material, only shear, and are therefore called *shear* waves.

Finally, it must be emphasized that although the longitudinal and transverse displacement fields individually satisfy the standard wave equation, the boundary conditions on the surface of a body must be applied to the complete displacement field (14-3). The boundary conditions will thus in general couple the longitudinal and transverse components, the only exception being plane waves in an infinitely extended medium.

Earthquake wave types: In earthquakes (see fig. 14.1) pressure waves are denoted P (for *primary*), because they arrive first due to the higher longitudinal phase velocity in any material. Typically the move at speeds of $4-7$ km/s in the Earth's crust. Shear waves move at roughly half the speed and thus arrive later at a seismometer. They are for this reason denoted by S (for *secondary*). In fluid material, such as the Earth's liquid core, shear waves cannot propagate. Besides these *body waves*, Earthquakes are also accompanied by *surface waves* to be discussed in section 14.3.

Harmonic analysis

A general mathematical theorem due to Fourier tells us that any time-dependent function may be resolved in a superposition of *harmonic* or *monochromatic* components, each oscillating with a single frequency. For linear differential equations — ordinary or partial — with time-independent coefficients this is particularly advantageous because it reduces the time-dependent problem to a time-independent one (for each frequency).

A real harmonic displacement field with *circular frequency* ω and *period* $2\pi/\omega$ satisfies the equation,

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\omega^2 \boldsymbol{u} \ . \tag{14-9}$$

The most general solution is a linear superposition of two time-independent *standing wave fields* $\boldsymbol{u}_1(\boldsymbol{x})$ and $\boldsymbol{u}_2(\boldsymbol{x})$,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_1(\boldsymbol{x}) \cos \omega t + \boldsymbol{u}_2(\boldsymbol{x}) \sin \omega t \ . \tag{14-10}$$

Instead of working with two real fields it is often most convenient to collect them in a single *complex* time-independent standing-wave field,

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_1(\boldsymbol{x}) + i\boldsymbol{u}_2(\boldsymbol{x}) \ . \tag{14-11}$$

The harmonic displacement field then becomes the real part of a complex field,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathcal{R}e \left[\boldsymbol{u}(\boldsymbol{x}) \, e^{-i\omega t} \right] \ . \tag{14-12}$$

The displacement velocity is correspondingly given by the imaginary part,

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t} = \omega \, \mathcal{I}m \left[\boldsymbol{u}(\boldsymbol{x}) \, e^{-i\omega t} \right] \ , \tag{14-13}$$

as may easily be verified.

Since the wave equation (14-2) is linear in $\boldsymbol{u}$, it is also satisfied by the velocity field $\partial \boldsymbol{u}/\partial t$ and thus by both the real and imaginary part of the complex field $\boldsymbol{u}(\boldsymbol{x})e^{-i\omega t}$, *i.e.* by the whole complex field itself. Inserting this field into the wave equation we obtain a single time-independent equation for the complex standing-wave field $\boldsymbol{u}(\boldsymbol{x})$,

$$\boxed{-\rho\omega^2 \boldsymbol{u} = \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{u} \ .} \tag{14-14}$$

eFreeHarmonicWave

It may be viewed as an *eigenvalue equation* for the operator $\mu\delta_{ij} \boldsymbol{\nabla}^2 + (\lambda+\mu)\nabla_i \nabla_j$ with eigenfunction $\boldsymbol{u}(\boldsymbol{x})$ and $-\rho\omega^2$ as eigenvalue. It may be shown that ω^2 is always real and positive (problem 14.4). In a finite body, the boundary conditions only allow solutions for a discrete set of eigenfrequencies, whereas in an infinite medium the eigenfrequencies normally form a continuum.

The harmonic analysis may immediately be extended to Navier's equation of motion with a time-dependent body force field $\boldsymbol{f}(\boldsymbol{x}, t)$. This will only add the complex harmonic amplitude $\boldsymbol{f}(\boldsymbol{x})$ of the force field to the right hand side of (14-14).

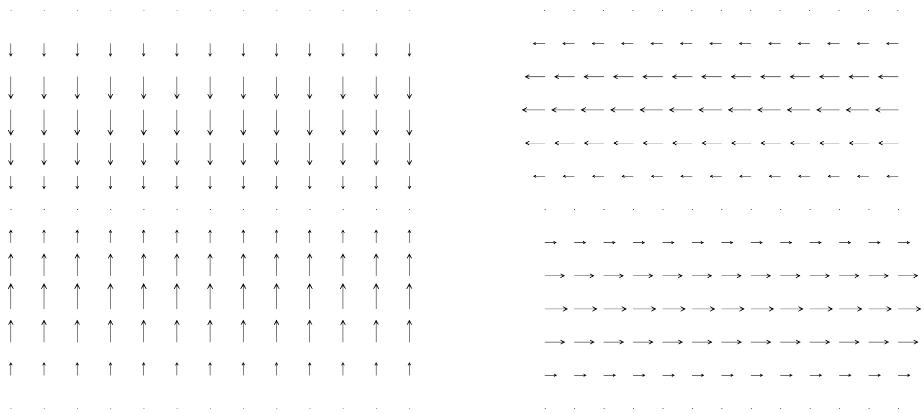
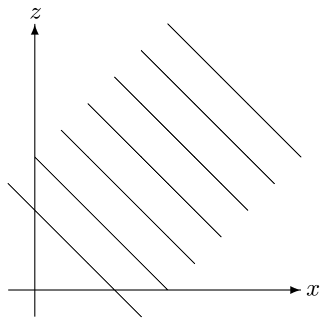


Figure 14.2: *Displacement fields for plane waves moving towards the top.* **(a)** *Longitudinal wave; the displacement oscillates in the direction of motion.* **(b)** *Transversal wave; the displacement oscillates orthogonally to the direction of motion.*



A plane wave has constant phase on planes orthogonal to the wave vector $\boldsymbol{k}$ *(here with* $k_y = 0$*). The phase is spatially periodic with wavelength* $\lambda = 2\pi/|\boldsymbol{k}|$.

Plane waves

Plane waves have infinite extension, and infinitely extended material bodies do not exist. Nevertheless, deeply inside a finite body, far from the boundaries, conditions are almost as if the body were infinite, and the displacement field may be resolved into a superposition of independent longitudinal or transverse plane waves. The condition for this to be possible is that the typical wave lengths contained in the wave should be much smaller than the dimensions of the body or the distance to boundaries.

It is instructive to carry through the harmonic analysis for a plane harmonic wave, described by (the real part of) a complex harmonic field of the form,

$$\boldsymbol{u} = \boldsymbol{a}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \ . \tag{14-15}$$

ePlaneDisplacementWave

Here $\boldsymbol{a}$ is the generally complex *amplitude* or *polarization vector*, $\boldsymbol{k}$ the *wave vector*, and ω the *circular frequency*. The wave's *direction of propagation* is $\boldsymbol{k}/|\boldsymbol{k}|$, its *wavelength* $2\pi/|\boldsymbol{k}|$, and its *period* $2\pi/\omega$. The *phase* of the wave is $\boldsymbol{k}\cdot\boldsymbol{x} - \omega t$, and its *phase velocity* $\omega/|\boldsymbol{k}|$. Inserting this field into (14-2) (or just $\boldsymbol{a}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ into (14-14)), we obtain,

$$\rho\omega^2\boldsymbol{a} = \mu\boldsymbol{k}^2\boldsymbol{a} + (\lambda+\mu)\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{a} \ . \tag{14-16}$$

ePlaneElasticWaveAmplitude

This is a simpel eigenvalue equation for the real symmetric 3×3 matrix $\mu\boldsymbol{k}^2\delta_{ij} + (\lambda+\mu)k_ik_j$, with eigenvector $\boldsymbol{a}$ and eigenvalue $\rho\omega^2$.

The eigenvectors are easily found. One is *longitudinal* with amplitude proportional to the wave vector itself, $\boldsymbol{a} \sim \boldsymbol{k}$. Inserting this into (14-16) we obtain $\rho\omega^2 = (\lambda+2\mu)\boldsymbol{k}^2$, showing that a general longitudinal harmonic plane wave is of the form (see fig. 14.2(a)),

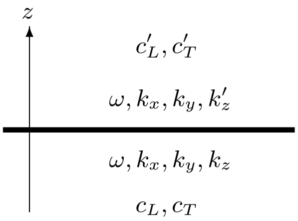
$$\boldsymbol{u}_L = A\,\boldsymbol{k}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\,, \qquad |\boldsymbol{k}| = \frac{\omega}{c_L}\,, \tag{14-17}$$

where c_L is given in (14-7) and A is an arbitrary complex number representing the longitudinal amplitude. The two other eigenvectors are *transverse* with amplitudes orthogonal to the wave vector, *i.e.* $\boldsymbol{k}\cdot\boldsymbol{a} = 0$, and it follows from (14-16) that $\rho\omega^2 = \mu\boldsymbol{k}^2$. The transverse harmonic plane wave is therefore of the form (see fig. 14.2(b)),

$$\boldsymbol{u}_T = \boldsymbol{a}_T\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\,, \qquad |\boldsymbol{k}| = \frac{\omega}{c_T}\,. \tag{14-18}$$

where $\boldsymbol{a}_T$ is an arbitrary vector orthogonal to $\boldsymbol{k}$. All of the transverse directions orthogonal to $\boldsymbol{k}$ propagate with the same phase velocity, and are thus *degenerate* eigenvectors. We may write $\boldsymbol{a}_T = A_1\boldsymbol{n}_1 + A_2\boldsymbol{n}_2$ where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are mutually orthogonal transverse vectors (both orthogonal to $\boldsymbol{k}$), and A_1 and A_2 are arbitrary complex numbers representing the transverse amplitudes.

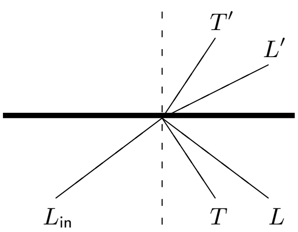
Fourier's theorem applied to both space and time variables tells us that the most general solution to the wave equation (14-2) is (the real part of) a superposition of longitudinal and transverse plane waves with different frequencies, directions of propagation, and amplitudes.



A plane interface between two media. The material properties are different on the two sides of the interface, but the frequency and the wave numbers components along the interface are the same.

14.2 Refraction and reflection

The simplest system which differs from from an infinitely extended medium consists in two semi-infinite media interfacing along a plane. The materials on the two sides of the interface are homogeneous and isotropic, but have different longitudinal and transverse phase velocities, c_L, c_T and c'_L, c'_T. A plane wave incident on one side of the interface will give rise to both a *refracted* wave on the other side and a *reflected* wave on the same side. Even if the incident wave is purely longitudinal or purely transverse, the refracted and reflected waves will in general be superpositions of longitudinal and transverse waves propagating in different directions. In this section we shall investigate some aspects of the these waves which even in this simplest non-trivial case are rather complicated.



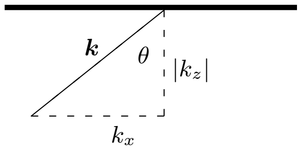
An incident longitudinal wave L_{in} is refracted into longitudinal and transverse components L', T', and also reflected into L, and T.

Snell's law

Taking the interface to be the xy-plane $z = 0$ of the coordinate system, the planar geometry is translationally invariant in all directions along x and y. That permits us to resolve the displacement field on either side into a superposition of plane waves of the form (14-15) where all the components have the same fixed values of ω, k_x and k_y on both sides of the interface, whereas in the z-direction the waves may have different values of k_z and k'_z. From this we conclude that the refracted and reflected waves propagate in the same plane as the incident wave.

In the following we shall without loss of generality choose the waves to propagate in the xz-plane with $k_y = 0$ and $k_x \geq 0$.

A simple geometric construction shows that the angle between the normal to the interface and the direction of propagation of any plane wave with phase velocity $c = \omega/|\boldsymbol{k}|$ is given by



Geometry for determining the angle between the direction of propagation and the normal to the interface.

$$\sin\theta = \frac{k_x}{|\boldsymbol{k}|} = \frac{k_x c}{\omega} \tag{14-19}$$

From the geometry it also follows that

$$|k_z| = k_x \cot\theta = \sqrt{\frac{\omega^2}{c^2} - k_x^2}\ . \tag{14-20}$$

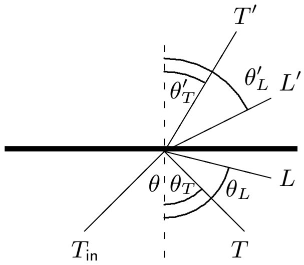
For $k_x < \omega/c$ the last expression is real and $\theta < 90°$. We shall later discuss what happens for $k_x > \omega/c$ where the squareroot becomes imaginary.

But since k_x and ω are the same for any plane wave component, the angles of incidence of two different wave components with phase velocities c_1 and c_2 must be related by *Snell's law*,

Willebrord van Roijen Snell (1580-1626). *Dutch mathematician. Contributed to geodesy (triangulation), and discovered the law of refraction.*

$$\boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{c_2}{c_1}}\ . \tag{14-21}$$

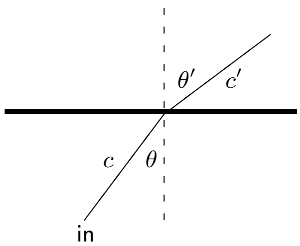
This relation applies to any combination of plane wave components whether they are longitudinal or transverse, on the same side (ipsilateral) as for reflection or on opposite sides (contralateral) as for refraction. Since $c_L > c_T$ we always have $\theta_L > \theta_T$ for the ipsilateral longitudinal and transverse components of a refracted or reflected wave. Reflected and incident waves of the same type will have the same angles with the normal. The angles of contralateral components are determined by the different material properties of the interfacing media, and cannot be generally characterized.



An incident transverse wave produces a reflected transverse wave having the same angle with the normal, $\Theta_T = \theta$, but may also produce a longitudinal wave with larger angle. Snell's law connects all these angles with the phase velocities.

> Snell's law takes the same form for elastic, acoustic, and electromagnetic waves. The fact that light moves with smaller phase velocity in water than in air immediately tells us that a light ray passing the plane water surface has a smaller angle with the normal in water than in air, thereby explaining the familiar observation that a straight rod apparently breaks when it is partially immersed into water. If the interface is curved we expect that Snell's law will be valid for wavelengths much smaller than the radii of curvature of the interface.

A peculiar thing happens when a refracted wave passes from lower to higher phase velocity, $c' > c$ (which it will always do from one side of the interface). Increasing the angle of incidence there will be a maximal incidence angle $\theta_{\max}$ satisfying $\sin\theta_{\max} = c/c'$ where the refraction angle becomes $\theta' = 90°$, and the wave appears to crawl along the interface. For $\theta > \theta_{\max}$ the incident wave is completely unable penetrate the interface and is totally reflected. Comparing with (14-20) *total reflection* is seen to correspond to imaginary values of the refracted



Refraction into a medium with higher phase velocity $c' > c$.

wave vector component k'_z. A similar phenomenon takes place for reflection in isotropic elastic media when the incident wave is transverse. The reflected longitudinal wave has always larger velocity than the reflected transverse wave, and there will be a maximal incident angle for longitudinal reflection. Beyond that angle, the reflected wave will necessarily be purely transverse.

Total reflection is well-known to divers looking at the water surface from below, or to fish looking at you from inside their aquarium. It is also of great importance for the functioning of optical fibers where total reflection guarantees that light sent down along the fiber stays inside the fiber even if it bends and winds.

Boundary conditions

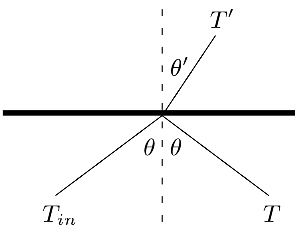
At the interface $z = 0$, the boundary conditions demand continuity of the displacement fields and the stress vectors on the two sides of the interface,

$$u'_x = u_x \ , \qquad u'_y = u_y \ , \qquad u'_z = u_z \ , \tag{14-22a}$$

$$\sigma'_{xz} = \sigma_{xz} \ , \qquad \sigma'_{yz} = \sigma_{yz} \ , \qquad \sigma'_{zz} = \sigma_{zz} \ . \tag{14-22b}$$

A single incident longitudinal or transverse wave can in principle generate one longitudinal and two transverse waves on either side of the interface. The amplitudes of the six waves are determined by the six boundary conditions. Intuitively it is fairly clear that "supertransverse" waves polarized orthogonally to the plane of incidence (*i.e.* along the y-direction) must decouple from the others which only involve the x and z-directions. We therefore only face four equations with four unknowns for the waves with polarization in the plane of incidence, or two equations with two unknowns for the "super-transverse" waves. It is still an unpleasant task to solve four equations with four unknowns, so in the remainder of this section we shall limit the analysis to a couple of cases resulting in only two equations with two unknowns.

"Supertransverse" waves at an interface



An incident supertransverse wave T_{in} is refracted and reflected into supertransverse waves, T' and T.

In a "supertransverse" wave the incident as well as the reflected and refracted components are polarized along $\boldsymbol{e}_y = (0, 1, 0)$. It is convenient to set $k_x = k$ and define $k_T = \sqrt{(\omega/c_T)^2 - k^2}$ and $k'_T = \sqrt{(\omega/c'_T)^2 - k^2}$. Leaving out the common factor $e^{i(kx-\omega t)}$, the only non-vanishing displacement components are,

$$u_y = e^{ik_T z} + Ae^{-ik_T z} \ , \qquad u'_y = A' e^{ik'_T z} \ , \tag{14-23}$$ eSuperTransverse

The first term in u_y represents the incident field, normalized to unity, while the second term represents the reflected wave with amplitude A. The field u'_y consists entirely of the refracted wave with amplitude A'. Since all the diagonal strains vanish, the boundary conditions are $u'_y = u_y$ and $\sigma'_{yz} = \sigma_{yz}$ at $z = 0$. Using that $\sigma_{yz} = \mu \nabla_z u_y$ and $\sigma'_{yz} = \mu' \nabla_z u'_y$ we are lead to the equations,

$$A' = 1 + A \ , \qquad \mu' k'_T A' = \mu k_T (1 - A) \ . \tag{14-24}$$

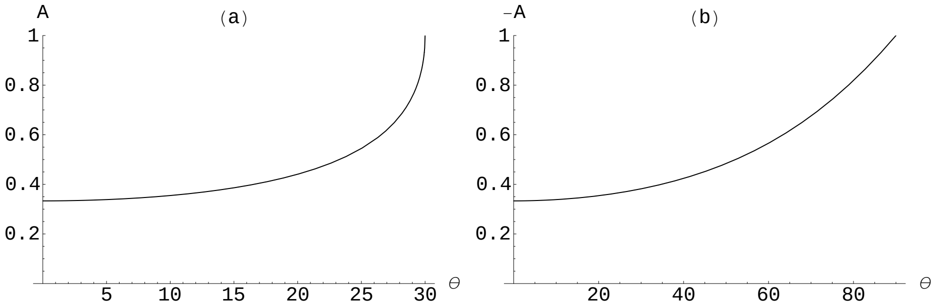


Figure 14.3: *"Supertransverse" waves at an interface with $\mu' = \mu$. The transmitted amplitudes are obtained by adding 1.* **(a)** *Reflected amplitude for $c'_T = 2c_T$. The maximum angle is 30° before total reflection sets in.* **(b)** *Reflected amplitude for $c'_T = \frac{1}{2}c_T$. The maximum angle is 90°. Notice that the amplitude is plotted as $-A$.*

The solution is,

$$A = \frac{\mu k_T - \mu' k'_T}{\mu k_T + \mu' k'_T} , \qquad A' = \frac{2\mu k_T}{\mu k_T + \mu' k'_T} , \tag{14-25}$$

and using that $k_T = k \cot\theta$ and $k'_T = k \cot\theta'$, the amplitudes may be expressed in terms of the angles and the ratio μ'/μ,

$$\boxed{A = \frac{\cot\theta - \frac{\mu'}{\mu}\cot\theta'}{\cot\theta + \frac{\mu'}{\mu}\cot\theta'} , \qquad A' = \frac{2\cot\theta}{\cot\theta + \frac{\mu'}{\mu}\cot\theta'} .} \tag{14-26}$$

Snell's law, $\sin\theta'/\sin\theta = c'_T/c_T$, connects the two angles, such that

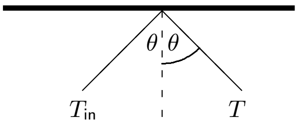
$$\cot\theta' = \sqrt{\frac{1}{\sin^2\theta'} - 1} = \sqrt{\left(\frac{c_T}{c'_T \sin\theta}\right)^2 - 1} . \tag{14-27}$$

In fig. 14.3 the reflected amplitude A is plotted as a function of the incident angle θ for two choices of material parameters.

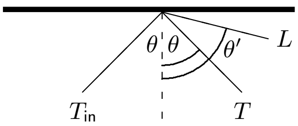
If $c'_T > c_T$ and $k > \omega/c'_T$, the refracted wave number becomes imaginary, $k'_T = i\kappa_T$ with $\kappa'_T = \sqrt{k^2 - (\omega/c'_T)^2}$. The refracted wave now decays with increasing z as $\exp(-\kappa'_T z)$ and only penetrates a finite distance into the upper half space. It has become a surface wave. The amplitudes are in this case,

$$A = \frac{\mu k_T - i\mu'\kappa'_T}{\mu k_T + i\mu'\kappa'_T} = e^{-i\phi} , \qquad A' = 1 + e^{-i\phi} , \tag{14-28}$$

where $\tan\frac{1}{2}\phi = \mu'\kappa'_T/\mu k_T$. Since the complex modulus is unity, $|A| = 1$, the totally reflected wave has the same intensity as the incident wave, although *phase shifted* by ϕ relative to the incident wave.



Reflection of a super-transverse wave at a free boundary.



Reflection of a transverse wave at a free boundary. The transverse reflection angle is always the same as the angle of incidence, whereas the longitudinal reflection angle is larger. There is a maximal value $\theta_{\max} = \arcsin q$ *for which a longitudinal reflection is possible.*

Reflection from a free surface

A longitudinal or transverse wave incident on a free surface can only be reflected, and the boundary conditions reduce in this case to the vanishing of the stress vector on the boundary $z = 0$,

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \; . \tag{14-29}$$

For the "supertransverse" wave (14-23) the solution is trivial. We simply take $A' = 0$ and find $A = 1$, such that $u_y \sim \cos(k_T z)$.

For a "normal" transverse wave the field is more complicated, because it can contain both longitudinal and transverse reflected components. Apart from an overall oscillating factor $e^{i(kx-\omega t)}$ the field is of the form,

$$\boldsymbol{u} = (-k_T, 0, k)e^{ik_T z} + A_T(k_T, 0, k)e^{-ik_T z} + A_L(k, 0, -k_L)e^{-ik_L z} \; , \tag{14-30}$$

eReflectionField

where A_T and A_L are the amplitudes of the reflected longitudinal and transverse fields, and where as before $k_T = \sqrt{(\omega/c_T)^2 - k^2}$ and $k_L = \sqrt{(\omega/c_L)^2 - k^2}$. It may readily be verified that the longitudinal field is a gradient, and the transverse fields have no divergence (remembering the oscillating factor $e^{i(kx-\omega t)}$).

From the above field we obtain the surface stresses (apart from the oscillating factor),

$$\sigma_{xz} = \frac{1}{2} i\mu \left((k^2 - k_T^2)(1 + A_T) - 2kk_L A_L \right) \; , \tag{14-31a}$$

$$\sigma_{zz} = i \left(2\mu k k_T (1 - A_T) + ((\lambda + 2\mu)k_L^2 + \lambda k^2)A_L) \right) \; . \tag{14-31b}$$

Using the relation $\rho\omega^2 = (\lambda + 2\mu)(k_L^2 + k^2) = \mu(k_T^2 + k^2)$, and requiring these stresses to vanish, we obtain the equations,

$$2kk_L A_L - (k^2 - k_T^2)A_T = k^2 - k_T^2 \; , \tag{14-32a}$$

$$(k^2 - k_T^2)A_L + 2kk_T A_T = 2kk_T \; , \tag{14-32b}$$

with the straightforward solution

$$A_T = \frac{4k^2 k_L k_T - (k^2 - k_T^2)^2}{4k^2 k_L k_T + (k^2 - k_T^2)^2} \; , \qquad A_L = \frac{4kk_T(k^2 - k_T^2)}{4k^2 k_L k_T + (k^2 - k_T^2)^2} \; , \tag{14-33}$$

Setting $k_T = k \cot\theta$ and $k_L = k \cot\theta'$, the solution may be cast into a convenient form depending only on the two angles,

$$\boxed{A_T = \frac{4\cot\theta\cot\theta' - (1 - \cot^2\theta)^2}{4\cot\theta\cot\theta' + (1 - \cot^2\theta)^2} \; , \qquad A_L = \frac{4\cot\theta(1 - \cot^2\theta)}{4\cot\theta\cot\theta' + (1 - \cot^2\theta)^2} \; .} \tag{14-34}$$

Snell's law $\sin\theta / \sin\theta' = c_T/c_L = q$, connects as before the two angles,

$$\cot\theta' = \sqrt{\frac{1}{\sin^2\theta'} - 1} = \sqrt{\frac{q^2}{\sin^2\theta} - 1} \; . \tag{14-35}$$

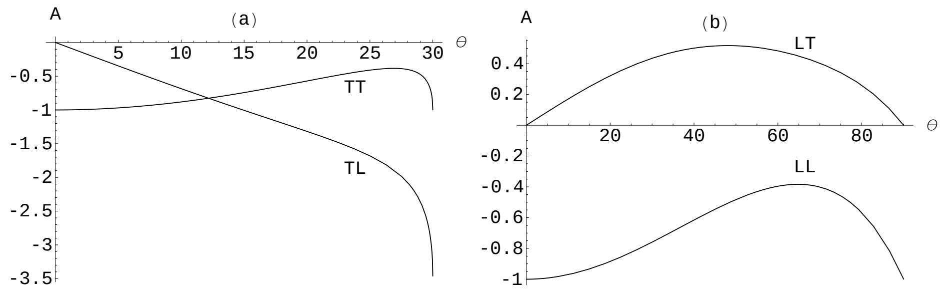
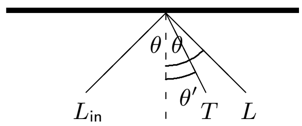


Figure 14.4: *Reflected amplitudes as function of the incident angle* θ *for* $q = \frac{1}{2}$. **(a)** *Transverse incident wave. The maximum angle with both transverse and longitudinal reflected waves is* 30°. **(b)** *Longitudinal incident wave. The maximum angle is in this case* 90°.

This clearly shows that a transverse incident wave produces a longitudinal reflected wave for $\sin\theta < q$, whereas for $\sin\theta > q$ only a reflected transverse wave is obtained.

The case of a longitudinal incident wave is very similar and is analyzed in problem 14.2. In fig. 14.4 the intensities of the reflected wave amplitudes are shown for both a transverse and longitudinal incident wave as a function of the angle of incidence.



Reflection of a longitudinal wave at a free boundary. The longitudinal reflection angle is the same as the angle of incidence, whereas the transverse reflection angle is smaller.

∗ 14.3 Surface waves

At a material interface there are special types of waves which do not penetrate into the bulk of the materials, but decay exponentially with the distance from the interface. We have already seen in the preceding section, how such wave components can arise in the refraction of an incident wave into a material with larger phase velocity when the angle of incidence becomes large enough. In this section we shall consider two kinds of free surface waves, *Rayleigh waves* and *Love waves*. Both have geophysical significance, in particular in relation to earthquakes where they arise at the encounter of seismic waves with the surface of the Earth.

Rayleigh waves

The most general exponentially decaying superposition of normal transverse and a longitudinal surface wave in the lower half space $z < 0$ is (apart from the oscillating factor $\exp(i(kx - \omega t))$ which is common to both terms),

$$\boldsymbol{u} = A_T(i\kappa_T, 0, k)e^{\kappa_T z} + A_L(k, 0, -i\kappa_L)e^{\kappa_L z} , \qquad (14\text{-}36)$$

John William Strutt, 3rd Baron Rayleigh (1842–1919). *Discovered and isolated the rare gas Argon for which he got the Nobel Prize (1904). Published the influential book "The Theory of Sound" on vibrations in solids and fluids in 1877-78.*

where A_T and A_L are generally complex constants. It is obtained from the general expression (14-30) by leaving out the incident wave (which diverges exponentially) and setting $k_T = i\kappa_T$ with $\kappa_T = \sqrt{k^2 - (\omega/c_T)^2}$, and $k_L = i\kappa_L$ with $\kappa_L = \sqrt{k^2 - (\omega/c_L)^2}$. Both κ_T and κ_L are real for $k > \omega/c_T$, because $c_T < c_L$. One

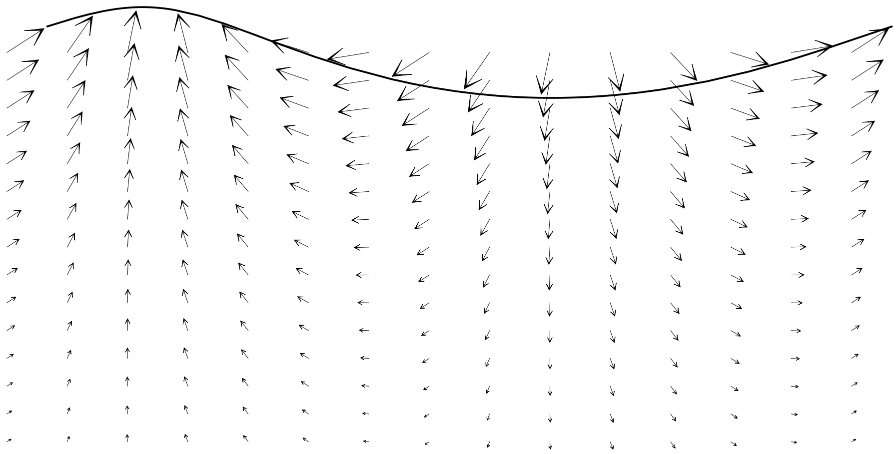


Figure 14.5: *One period of a Rayleigh wave moving to the right. The shear component creates a wave-like motion of the surface resembling a water wave (except they are incompressible). Notice the exponential decay of the wave under the surface.*

may verify directly that the longitudinal wave is indeed a gradient field, and that the transverse wave is free of divergence.

The free surface boundary conditions are the same as in the preceding section $\sigma_{xz} = \sigma_{zz} = 0$ for $z = 0$, and leaving out the terms due to the incident wave on the right hand side of (14-32), we find from the left hand side,

$$2ik\kappa_L A_L - (k^2 + \kappa_T^2)A_T = 0 \ , \tag{14-37}$$

$$(k^2 + \kappa_T^2)A_L + 2ik\kappa_T A_T = 0 \ . \tag{14-38}$$

Since $iA_L/A_T = (k^2 + \kappa_T^2)/2k\kappa_L = 2k\kappa_T/(k^2 + \kappa_T^2)$, these equations only have a non-vanishing solution for,

$$(k^2 + \kappa_T^2)^2 = 4k^2\kappa_T\kappa_L \ . \tag{14-39}$$

Defining the phase velocity along the surface $c = \omega/k$, this condition turns into an equation for c,

$$\boxed{\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\sqrt{\left(1 - \frac{c^2}{c_T^2}\right)\left(1 - \frac{c^2}{c_L^2}\right)} \ .} \tag{14-40}$$

The simplest way to solve this equation is to square it and isolate the ratio $q = c_T/c_L$ in terms of the ratio $\xi = c/c_T$,

$$q = \sqrt{\frac{16 - 24\xi^2 + 8\xi^4 - \xi^6}{16(1 - \xi^2)}} \ . \tag{14-41}$$ eRayleighPhaseVelocity

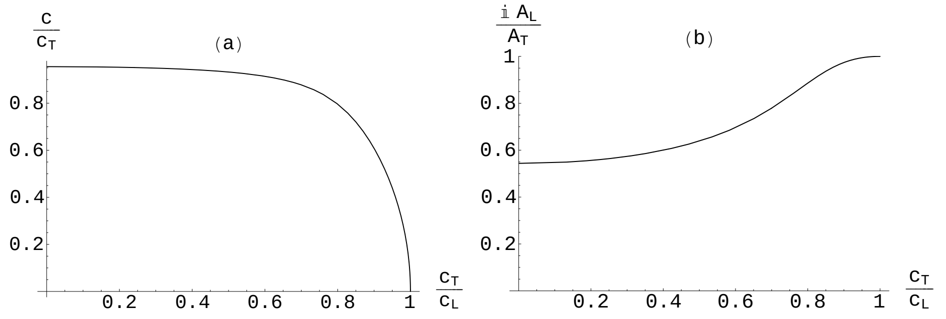


Figure 14.6: *Rayleigh waves.* **(a)** *Phase velocity as a function of* $q = c_T/c_L$. *For all possible physical values,* $q < 1/\sqrt{2} \approx 0.7$, *the phase velocity is nearly equal to the phase velocity of transverse waves.* **(b)** *Ratio of longitudinal and transverse amplitudes as a function of* q.

In fig. 14.6(a) the phase velocity ξ of Rayleigh waves has been plotted as a function of q. The maximal value $\xi_0 = 0.955313\ldots$ is the real root of the polynomial in the numerator under the squareroot.

Typical values of q are around 0.5, showing that the value of ξ is close to unity in all practical cases. Expanding (14-41) to lowest order near $\xi = 1$ we find the approximation $\xi = 1 - 1/2(11 - 16q^2)$ which for $q = 0.5$ is better than 1%. The phase velocity of Rayleigh waves is thus normally just a little below the phase velocity of free transverse waves.

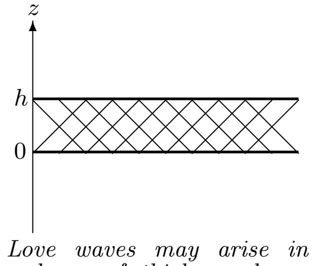
Seismic waves created deep inside the Earth's crust are reflected from the surface. If the angle of incidence is large enough, the transverse components will excite Rayleigh waves running along the surface. Since their speed is slightly lower than the transverse waves, they arrive even later than S-waves at a seismometer (if they originate in the same point). During the passing of a Rayleigh wave, the surface suffers a combination of compressional and vertical shear displacements, much like a wave rolling across the sea (see fig. 14.5 and fig. 14.1). Horizontal shear displacements are absent in a Rayleigh wave.

Love waves

Augustus Edward Hough Love (1863–1940). *British scholarly physicist. Contributed to the mathematical theory of elasticity, and to the understanding and analysis of the waves created by earthquakes.*

One could think that there might be "supertransverse" free surface waves, either at a free surface or at a material interface, but neither of these types are in fact possible (see problem 14.3). Love found, however, in 1911 that supertransverse free waves may be created if the surface material is heterogeneous with elastic properties that change with height z.

The simplest geometry is obtained by placing a layer of a material of thickness h situated on top of a material filling the half-space $z < 0$. Under the conditions that $c_T > c'_T$ and $\omega/c_T < k < \omega/c'_T$ there will be a solution which is exponentially damped in the lower half-space and has running waves in the upper layer.



Love waves may arise in a layer of thickness h *on top of a half-space* $z < 0$ *of other material.*

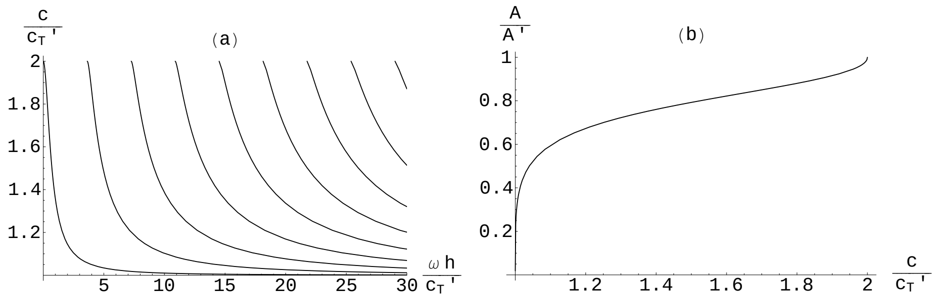


Figure 14.7: *Love waves for $c'_T = 0.5c_T$ and $\mu' = \mu$.* **(a)** *Phase velocity c/c'_T as a function of $\omega h/c'_T$. For a given frequency there is a finite number of possible phase velocities.* **(b)** *Amplitude ratio A/A' as a function of phase velocity c/c'_T.*

Anticipating that the stress, $\sigma'_{yz} = \mu' \nabla_z u'_y$, must vanish at the top of the layer, $z = h$, a supertransverse field takes the following form in the two media,

$$u_y = Ae^{\kappa_T z} , \qquad u'_y = A' \cos k'_T (h - z) , \tag{14-42}$$

where $\kappa_T = \sqrt{k^2 - (\omega/c_T)^2} = k\sqrt{1 - (c/c_T)^2}$ and $k'_T = \sqrt{(\omega/c'_T)^2 - k^2} = k\sqrt{(c/c'_T)^2 - 1}$. The remaining boundary conditions are as before $u'_y = u_y$ and $\sigma'_{yz} = \sigma_{yz}$ at $z = 0$, leading to

$$A = A' \cos(k'_T h) , \qquad \mu\kappa_T A = \mu' k'_T A' \sin(k'_T h) . \tag{14-43}$$

A non-trivial solution can only exist for,

$$\mu\kappa_T = \mu' k'_T \tan(k'_T h) . \tag{14-44}$$

Introducing $c = \omega/k$ we have $\kappa_T = k\sqrt{1 - (c/c_T)^2}$ and $k'_T = k\sqrt{(c/c'_T)^2 - 1}$. Solving the above equation for kh, we obtain

$$kh = \frac{1}{\sqrt{\frac{c^2}{c'^2_T} - 1}} \arctan \frac{\mu\sqrt{1 - \frac{c^2}{c_T^2}}}{\mu'\sqrt{\frac{c^2}{c'^2_T} - 1}} . \tag{14-45}$$

For any value of the phase velocity in the interval $c'_T < c < c_T$ this permits us to calculate the value of kh and — since h is assumed known — of $\omega = ck$. Conversely, for a given frequency ω one may solve this equation for the allowed values of c. The infinity of branches of inverse tangent yields an infinite number of possible frequencies for a given phase velocity. Conversely, there is only a finite number of allowed phase velocities for a given frequency. The solutions c/c'_T are plotted in fig. 14.7(a) as functions of the dimensionless frequency parameter $\omega h/c'_T$ for a choice of material parameters. In fig. 14.7(b) the amplitude ratio A/A' is plotted as a function of the phase velocity parameter c/c'_T.

A surface layer thus acts like a wave guide for Love waves. Contrary to Rayleigh waves Love waves are *dispersive* with phase velocity that depends on the wavelength (or frequency). Since $c'_T < c < c_T$ Love waves move faster than Rayleigh waves in the surface layer, but slower than shear waves in the bulk. Love waves thus arrive before the Rayleigh waves originating in the same point (see fig. 14.1). In earthquakes Love waves are the most destructive because of the shearing motion of the surface layer which is not very well tolerated by buildings.

Problems

14.1 Derive the form of Navier's equation of motion when the Lamé coefficients depend on position.

14.2 Show that the reflection amplitudes for a longitudinal incident wave on a free surface are,

$$A_L = \frac{4\cot\theta\cot\theta' - (1-\cot^2\theta')^2}{4\cot\theta\cot\theta' + (1-\cot^2\theta')^2}\ , \quad A_T = -\frac{4\cot\theta(1-\cot^2\theta')}{4\cot\theta\cot\theta' + (1-\cot^2\theta')^2} \tag{14-46}$$

with $\cot\theta' = \sqrt{-1+1/(q\sin\theta)^2}$.

14.3 Show that supertransverse Love waves cannot exist at an interface (or at a free surface).

∗ **14.4** Show that the eigenvalues ω^2 of the amplitude equation for free waves (14-14) are real and positive when the boundary conditions specify the vanishing of the displacement field or of the stress vector.

∗ **14.5** **(a)** Show that an arbitrary vector field may be resolved into (not necessarily unique) longitudinal and transverse components, and that the longitudinal component may be chosen to be a gradient. **(b)** Show that the individual components of a mixed field (14-3) may always be chosen to satisfy Navier's equation of motion individually.

