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QFT takehome final

1) Zero-dimensional field theory of N-color "quarks"

fields

"quarks" ψ_k, ψ^k (not fermionic)

"gluons" A_k^ℓ $k, \ell = 1, 2, \cdots, N$

action

$$S = -\frac{N}{2}\mu^2 A^2 + \bar{\psi}(gA - m)\psi$$

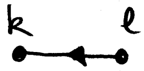
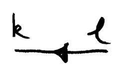
$$= -\frac{N}{2}\mu^2 \sum_{k,\ell=1}^{N} A_k^\ell A_\ell^k + \sum_{k,\ell=1}^{N} \psi_k A_\ell^k \psi^\ell - m \sum_{k=1}^{N} \psi_k \psi^k$$

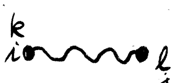
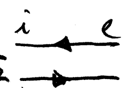
path integral

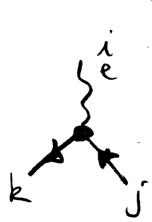
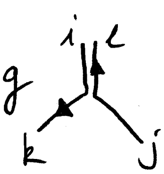
$$Z[J, \eta, \bar{\eta}] = \int [dA\, d\bar{\psi}\, d\psi]\; e^{+S[A,\bar{\psi},\psi] + J\cdot A + \bar{\psi}\cdot\eta + \bar{\eta}\cdot\psi}$$

Note: no space dependence, QFT at a single point, hence "zero-dimensional".

(a) derive Feynman rules:

"quark"  $= \frac{1}{m}$  $= \frac{1}{m}\delta_k^\ell$

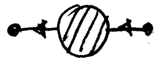
"gluon"  $= \frac{1}{N\mu^2}$  $= \frac{1}{N\mu^2}$

"quark-gluon" vertex  $= -g$  $= -g\,\delta_k^i \delta_\ell^j$

(warning: these might have errors – trust your own)

1, continued

(b) Integrate out $\bar{\psi}, \psi$. Write the path integral for the "quark" propagator in background "gluon" field

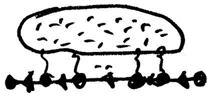
$G_k^{\ell} =$  $= \frac{1}{Z} \int [dA] \left(\frac{1}{m - gA} \right)_k^{\ell} \cdots$

$= \left\langle \frac{1}{m - gA} \right\rangle$

(c) Use O(N) symmetry to show that

$$G_k^{\ell} = \delta_k^{\ell} G$$

(d) Show that

 $= \sum_{2n=0}^{\infty}$ 

(e) list all 1- and 2-gluon Feynman graphs (optionally also 3-gluon graphs). Draw them from now on in the upper-half plane (above the quark line)

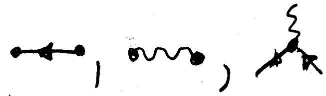
(f) Evaluate them. (Simple;  $= \delta_{\ell}^{\ell} = N$)

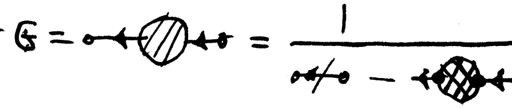
(g) Which ones dominate for $N \to \infty$? Which ones are subdominant?

(h) Insert a gluon line in any allowable way into a planar diagram. Which inserts are subdominant in $N \to \infty$ limit?

(i) Which Feynman diagrams survive in $N \to$ limit?

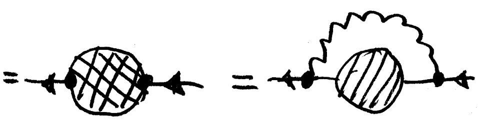
2) Planar field theory

Consider  Feynman diagrams that can be drawn in (upper-half) plane

Connected propagator $G =$  $= \frac{1}{m - \Sigma}$

1-P-I propagator: $\Sigma =$ 

(a) show that for planar graphs in zero-dimensional case, problem 1

$\Sigma =$  $= \frac{g^2}{\mu^2} G$

(b) show that

$$G(m) = \frac{\mu^2}{2}\left(m - \sqrt{m^2 - \frac{4g^2}{\mu^2}}\right)$$

(c) explain why $+\sqrt{\cdots}$ solution is not good

Remark: With substitution $m \to$ energy E

$gA \to$ hamiltonian H

the resulting formula is a famous 1955 law used in nuclear physics, atomic physics, and condensed matter theory.

(d) bonus: what 1955 law??

problem 3).

Consider the matrix $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

(a) Show that γ^5 anticommutes with each of the γ^μ matrices, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

(b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

(c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

(d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

(e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ and γ^5.

Conventions: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$, $\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$,
$\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \frac{1}{6}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda)$,
and ditto for the $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}$.

next: problem 4).

Casimir energy Consider a massless real scalar field ϕ in two dimensions, confined to a box of length L with antiperiodic boundary conditions. This is, a field which satisfies $\phi(t, L) = -\phi(t, 0)$.

(a) Find the Fourier mode expansion for the field ϕ.

(b) Remember that the the Feynman Green's function for a field in flat space is given by

$$< 0|T(\phi(x)\phi(x'))|0 >= G_F(x, x')_0 = \int d^2k \exp(ik(x-x'))/k^2 \times \left(\frac{\mathbf{1}}{\mathbf{2}\pi}\right)^2$$

Calculate $G_F(x, x')$ as a function of it's arguments.

(c) For the case at hand with the given boundary conditions, a Feymann Green's function is given by

$$G_F(x,x') = \sum_k \int dk^0 \exp(ik(x-x'))/k^2$$

where k runs over the allowed set of modes from the mode expansion. Show that this is equal to a sum over images

$$G_F(x,x') = \sum_{n=-\infty}^{\infty} (-1)^n G_F(x+nL,x')_0$$

of the Green's function for flat space.

(d) Show that the Hamiltonian density in terms of the field ϕ is given by

$$\mathcal{H}(x,t) = \frac{1}{2}[\dot{\phi}(x,t)^2 + \phi_{,x}(x,t)^2]$$

(e) Consider a point splitting Hamiltonian given by

$$\mathcal{H}(x,t)_\epsilon = \frac{1}{2}[\dot{\phi}(x,t)\dot{\phi}(x+\epsilon,t) + \phi_{,x}(x,t)\phi_{,x}(x+\epsilon,t)]$$

Calculate

$$< 0|\mathcal{H}(x,t)_\epsilon|0 >$$

both for the theory in infinite space and in the theory with boundary conditions.

(f) Show that the difference between the two results gives a finite result in the limit

$$\lim_{\epsilon\to 0}([\mathcal{H}_\epsilon]_{box} - [\mathcal{H}_\epsilon]_{flat})$$

What is the final value?

Consider a theory of a single Dirac spinor with two kinds of mass terms,

problem 5)

$$\mathcal{L} = \frac{1}{2}(i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - im'\bar{\psi}\gamma_5\psi)$$

a) Show that the derivative term is invariant under the *chiral transformation*

$$\psi \to e^{i\alpha\gamma_5}\psi \ ,$$

for arbitrary constant α.

b) Use such a chiral transformation to transform the pseudo-scalar mass m' away. What is the mass of the resultant Dirac field?