(a) :

$\gamma^\mu\gamma^\nu = \pm\gamma^\nu\gamma^\mu$ where the sign is '+' for $\mu=\nu$ and '−' otherwise. Hence for any product Γ of the γ matrices, $\gamma^\mu\Gamma = (-1)^{n_\mu}\Gamma\gamma^\mu$ where n_μ is the number of $\gamma^{\nu\neq\mu}$ factors of Γ. For $\Gamma=\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $n_\mu=3$ for any $\mu=0,1,2,3$; thus $\gamma^\mu\gamma^5=-\gamma^5\gamma^\mu$.

(b) :

First,

$$\begin{aligned}\left(\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = +i\gamma^3\gamma^2\gamma^1\gamma^0\\ &= +i((\gamma^3\gamma^2)\gamma^1)\gamma^0 = (-1)^3 i\,\gamma^0((\gamma^3\gamma^2)\gamma^1)\\ &= (-1)^{3+2} i\,\gamma^0(\gamma^1(\gamma^3\gamma^2)) = (-1)^{3+2+1} i\,\gamma^0(\gamma^1(\gamma^2\gamma^3))\\ &= +i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv +\gamma^5.\end{aligned} \tag{S.1}$$

Second,

$$\begin{aligned}(\gamma^5)^2 &= \gamma^5(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) = -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0\\ &= +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = -\gamma^0(\gamma^1\gamma^1)\gamma^0 = +\gamma^0\gamma^0 = +1.\end{aligned} \tag{S.2}$$

(c) :

Any four distinct γ^κ, γ^λ, γ^μ, γ^ν are γ^0, γ^1, γ^2, γ^3 in some order. They all anticommute with each other, hence $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu=\epsilon^{\kappa\lambda\mu\nu}\gamma^0\gamma^1\gamma^2\gamma^3\equiv -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$. The rest is obvious.

(d) :

$$\begin{aligned}i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5 &= \gamma_\kappa\,\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}\\ &= \tfrac14\,\gamma_\kappa\left(\gamma^\kappa\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} - \gamma^{[\lambda)}\gamma^\kappa\gamma^{(\mu}\gamma^{\nu]} + \gamma^{[\lambda}\gamma^{\mu)}\gamma^\kappa\gamma^{(\nu]} - \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}\gamma^\kappa\right)\\ &= \tfrac14\left(4\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} + 2\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} + 4g^{[\lambda\mu}\gamma^{\nu]} + 2\gamma^{[\nu}\gamma^\mu\gamma^{\lambda]}\right)\\ &= \tfrac14(4+2+0-2)\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}.\end{aligned} \tag{S.3}$$

(e) ___________:

Proof by inspection: In the Weyl basis, the 16 matrices are

$$\begin{aligned} \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ i\gamma^{[i}\gamma^{j]} &= \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \qquad i\gamma^{[0}\gamma^{i]} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & +i\sigma^i \end{pmatrix}, \\ \gamma^5\gamma^0 &= \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \qquad \gamma^5\gamma^1 = \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \end{aligned} \tag{S.4}$$

and their linear independence is self-evident. Since there are only 16 independent 4×4 matrices altogether, any such matrix Γ is a linear combination of the matrices (S.4). $\mathcal{Q.E.D.}$

Algebraic Proof: Without making any assumption about the matrix form of the γ^μ operators, let us consider the Clifford algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. Because of these anticommutation relations, one may re-order any product of the γ's as $\pm\gamma^0\cdots\gamma^0\,\gamma^1\cdots\gamma^1\,\gamma^2\cdots\gamma^2\,\gamma^3\cdots\gamma^3$ and then further simplify it to $\pm(\gamma^0 \text{ or } 1) \times (\gamma^1 \text{ or } 1) \times (\gamma^2 \text{ or } 1) \times (\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the 16 operators 1, γ^μ, $i\gamma^{[\mu}\gamma^{\nu]}$, $-i\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \epsilon^{\lambda\mu\nu\rho}\gamma^5\gamma_\rho$ (*cf.* (d)) or $i\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = \epsilon^{\kappa\lambda\mu\nu}\gamma^5$ (*cf.* (c)). Consequently, any operator Γ algebraically constructed of the γ^μ's is a linear combination of these 16 operators.

Incidentally, the algebraic argument explains why the γ^μ (and hence all their products) should be realized as 4×4 matrices since any lesser matrix size would not accommodate 16 independent products. That is, the γ's are 4×4 matrices in four spacetime dimensions; different dimensions call for different matrix sizes. Specifically, in spacetimes of *even* dimensions d, there are 2^d independent products of the γ operators, so we need matrices of size $2^{d/2} \times 2^{d/2}$: 2×2 in two dimensions, 4×4 in four, 8×8 in six, 16×16 in eight, 32×32 in ten, *etc.*,*etc.*.

In odd dimensions, there are only 2^{d-1} independent operators because $\gamma^{d+1} \equiv (i)\gamma^0\gamma^1\cdots\gamma^{d-1}$ — the analogue of the γ^5 operator in 4d — commutes rather than anticommutes with all the γ^μ and hence with the whole algebra. Consequently, one has two distinct representations of the Clifford algebra — one with $\gamma^{d+1} = +1$ and one with $\gamma^{d+1} = -1$ — but in each representation there are only 2^{d-1} independent operator products, which call for the matrix size of $2^{(d-1)/2} \times 2^{(d-1)/2}$. For example, in three spacetime dimensions (two space, one time), can take $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, i\sigma_2)$ for $\gamma^4 \equiv i\gamma^0\gamma^1\gamma^2 = +1$ or $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, -i\sigma_2)$ for $\gamma^4 = -1$, but in both cases we have 2×2 matrices. Likewise, we have 4×4 matrices in five dimensions, 8×8 in 7D, 16×16 in 9D, 32×32 in 11D, *etc.*, *etc.*

a) The field operator $\phi(x,t)$ must satisfy antiperiodic boundary conditions.

$$\phi(x+L,t) = -\phi(x,t) = e^{i\pi}\phi(x,t)$$

suggesting that all modes must have wavenumbers of the form

$$k_n = \frac{\pi}{L}(2n+1)$$

problem 4)

for some integer n. Thus, we have a mode expansion

$$\phi(x,t) = \sum_n \left[e^{ik_n(x-t)}a(k_n) + e^{-ik_n(x-t)}a^\dagger(k_n)\right]$$

b) The Green's function is given by

$$\begin{aligned}
G_F(x,x') &= \int \frac{dk\, d\omega}{(2\pi)^2} \frac{e^{-i\omega\Delta t}e^{ik\Delta x}}{k^2-\omega^2} \\
&= i\int \frac{dk\, d\omega}{(2\pi^2)} \frac{e^{i(\omega\Delta\tau + k\Delta x)}}{k^2+\omega^2} \\
&= i\int_0^\infty d\alpha \int \frac{dk}{2\pi} e^{-\alpha k^2 + ik\Delta x} \int \frac{d\omega}{2\pi} e^{-\alpha\omega^2 + i\omega\Delta\tau} \\
&= i\int_0^\infty \frac{d\alpha}{4\pi\alpha} e^{-\frac{1}{4\alpha}[(\Delta\tau^2)+(\Delta x)^2]} \\
&= -i\int_0^\infty \frac{du}{4\pi u} e^{-u[(\Delta\tau^2)+(\Delta x)^2]}
\end{aligned}$$

I got the second line by Wick rotating (Peskin and Shroeder, p. 193 or Srednicki, p. 216) and the line after that by using the identity

$$\frac{1}{B} = \int_0^\infty d\alpha\, e^{-\alpha B}$$

The integral on the last line is formally divergent, but note that

$$-\frac{\partial}{\partial B}\int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha B} = \int_0^\infty d\alpha\, e^{-\alpha B}$$

in order to recover

$$\begin{aligned}
G_F(x,x') &= -\frac{i}{4\pi}\ln\left[(\Delta\tau)^2 + (\Delta x)^2\right] \\
&= -\frac{1}{2\pi}\ln|x-x'|
\end{aligned}$$

after Wick rotating things back.

c) Start with

$$\begin{aligned}
G_F(x,x') &= \sum_{k_n}\int \frac{d\omega}{2\pi}\frac{e^{i(k_n x - \omega t)}}{k_n^2-\omega^2} \\
&= \int \frac{d^2k}{(2\pi)^2}\frac{e^{ikx}}{k^2}\sum_n (2\pi)\delta\left(k-(2n+1)\tfrac{\pi}{L}\right) \\
&= \sum_m \int \frac{dk\, d\omega}{(2\pi)^2}\frac{e^{ikx}}{k^2}e^{i\pi m}e^{imkL} \\
&= \sum_m (-1)^m G_F(x+mL, x')
\end{aligned}$$

using the Poisson sum formula on the third line.

d) The Lagrangian for the theory is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

with canonical momentum $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ so the Hamiltonian is

$$\mathcal{H} = \Pi \dot{\phi} - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi_{,x}^2$$

e) The point-splitting Hamiltonian is

$$\mathcal{H}_\epsilon \equiv \frac{1}{2}\dot{\phi}(x,t)\dot{\phi}(x+\epsilon,t) + \frac{1}{2}\phi_{,x}(x,t)\phi_{,x}(x+\epsilon,t)$$

so

$$\langle 0|\mathcal{H}_\epsilon(x,t)|0\rangle = -\frac{1}{2}\left(\partial_0^2 - \partial_x^2\right)G_F(x, x+\epsilon)$$

In flat space, this is simply

$$= -\frac{1}{2\pi}\frac{1}{\epsilon^2}$$

and in the box it is

$$-\frac{1}{2\pi}\frac{1}{\epsilon^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(nL)^2}$$

As expected, both Hamiltonians diverge in the $\epsilon \to 0$ limit.
e) Subtracting the flat Hamiltonian from the box Hamiltonian eliminates the ϵ dependence, leaving a difference of

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(nL)^2} = \frac{1}{\pi L^2}\left(\sum_{n=1}\frac{1}{n^2} - 2\sum_{n=1}\frac{1}{(2n)^2}\right) = \frac{\pi}{12L^2}$$

The Lagrangian is

problem 5) solution

$$\mathcal{L} = \frac{1}{2}\left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi - im'\bar{\psi}\gamma_5\psi\right)$$

a) We perform a chiral transformation

$$\psi \to e^{i\alpha\gamma_5}\psi$$
$$\bar{\psi} \to \psi^{\dagger}e^{-i\alpha\gamma_5}\gamma^0 = \bar{\psi}e^{i\alpha\gamma_5}$$

where we can anti-commute every power of γ_5 in the exponential past the γ_0 in the definition of $\bar{\psi}$. Then the derivative term in the Lagrangian becomes

$$\bar{\psi}e^{i\alpha\gamma_5}\gamma^{\mu}\partial_{\mu}e^{i\alpha\gamma_5}\psi = \bar{\psi}\gamma^{\mu}\partial_{\mu}e^{-i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi = \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$$

b) Transforming the mass terms gives

$$m\bar{\psi}e^{2i\alpha\gamma_5}\psi + im'\bar{\psi}e^{2i\alpha\gamma_5}\gamma_5\psi$$

Fortunately, we can use $(\gamma_5)^2 = 1$ to simplify the exponential.

$$e^{2i\alpha\gamma_5} = \sum_{n=0}^{\infty}\left[\frac{(2i\alpha)^2 n}{(2n)!} + \gamma_5\frac{(2i\alpha)^{2n+1}}{(2n+1)!}\right] = \cos 2\alpha + i\gamma_5\sin 2\alpha$$

so the sum of the mass terms is

$$\bar{\psi}\Big[\left(m\cos 2\alpha - m'\sin 2\alpha\right) + i\gamma_5\left(m'\cos 2\alpha + m\sin 2\alpha\right)\Big]\psi$$

Rotating the chiral mass term away requires finding a value of *alpha* such that

$$m' \cos 2\alpha + m \sin 2\alpha = 0$$

which is satisfied by

$$\frac{m'}{m} = -\tan 2\alpha$$

so the new mass is given by

$$\sqrt{m^2 + m'^2}$$