Periodic-Orbit Quantization of Chaotic Systems

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We demonstrate the utility of the periodic-orbit description of chaotic motion by computing from a few periodic orbits highly accurate estimates of a large number of quantum resonances for the classically chaotic three-disk scattering problem. The symmetry decompositions of the eigenspectra are the same for the classical and the quantum problem, and good agreement between the periodic-orbit estimates and the exact quantum poles is observed.

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It is a characteristic feature of dynamical systems of a few degrees of freedom that the motion is often organized around a few *fundamental* cycles. These short cycles capture the skeletal topology of the motion in the sense that any long orbit can approximately be pieced together from the fundamental cycles. Moreover, many quantities of interest can be computed as averaged over periodic orbits. In Ref. 1 a highly convergent expansion around short cycles has been introduced and applied to evaluation of classical chaotic averages. The goal of this Letter is to demonstrate that the curvature expansions[1] of periodic-orbit sums[2–4] are an equally powerful tool for evaluation of *quantum* resonances of classically chaotic systems.

In this approach, the averages over chaotic dynamical systems are determined from the zeros of dynamical ζ functions,[5] defined as expansions of infinite products of the form

$$\zeta^{-1}=\prod_p(1-t_p)=1-\sum_f t_f-\sum_p c_p\,, \qquad (1)$$

with weight t_p associated to every primitive (nonrepeating) periodic orbit (or *cycle*) p. The key observation is that the expanded product allows a regrouping of terms into dominant *fundamental* contributions and decreasing *curvature* corrections. Computations with ζ functions are rather straightforward; typically one determines lengths and stabilities of a finite number of shortest periodic orbits, substitutes them into (1), and estimates the zeros of (1) from such polynomial approximations.

We shall apply here the expansion (1) to evaluation of repeller escape rates. The classical repeller escape rate γ is determined[1,6,7] by the largest zero of $1/\zeta(s)$ (s real) with each prime cycle weighted by

$$t_p(s)=\Lambda_p^{-1}e^{-sT_p}\,. \qquad (2)$$

Here T_p is the period of the prime cycle p and $\mu_p=\ln(\Lambda_p)$ is its stability exponent, where Λ_p is the leading eigenvalue of the cycle Jacobian. The associated *quantum* amplitude is essentially the square root of the classical weight. This follows from the stationary phase formula[2–4] for determining the poles of the scattering matrix in terms of cycles, rewritten[8,9] as the logarithmic derivative of the infinite product of ζ functions $Z(k)=\prod_{j=0}^{\infty}\zeta_j^{-1}(k)$, where the weights of the prime cycles for the different ζ_j's are

$$t_p^{(j)}=\exp[-\mu_p(\tfrac{1}{2}+j)+(i/\hbar)S_p(k)+i\pi\nu_p/2]\,, \qquad (3)$$

where $S_p(k)$ is the action and ν_p is the Maslov index (in the three-disk example considered below, k is the quantum wave number). The zeros of $Z(k)$ in the complex k plane determine the eigenvalues or resonances of the quantum system; here we shall compute only those closest to the real energy axis, which are given by the zeros of $1/\zeta_0(k)$.

As it stands, the Euler product (1) is a product over an infinity of prime cycles or arbitrary length, and its utility as a computational tool is far from obvious. It is one of many formally equivalent cycle-averaging expressions, and its connection to the Gutzwiller periodic-orbit sum[2] has been known for some time.[8–10] What has not be recognized is that the intuition derived from classical chaotic dynamics[1,7] singles out one particular expansion, exploiting the fact that long periodic orbits can be approximated by short ones.

More precisely, there are two types of contributions to the curvature expansion (1): a set t_f of *fundamental* periodic orbits and an infinite series of *curvature* corrections c_p. The fundamental cycles have no shorter approximants; they are the "simplest" cycles in the sense that all longer orbits can be pieced together from the fundamental cycles as fundamental building blocks. The curvatures are *differences* of long orbits and their estimates based on shorter orbits [see Eq. (5) below]. They account for corrections as one resolves the dynamics on longer and longer times, i.e., finer and finer resolution in phase space. In averages dominated by positive entropy of unstable orbits, these differences decay[1,7] *exponen-*

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tially and the curvature expansions are expected to be highly convergent.

Here we shall illustrate the convergence of curvature expansions by computing the classical escape rates and quantum resonances for scattering off three disks[11] (we refer the reader to Refs. 9, 12, and 13 for detailed discussions). In this model the classical motion can be visualized as a pinball bouncing in a plane between three equally spaced disks of equal radius, and the quantum dynamics is described by wave functions which vanish on the boundaries of the disks. For billiard motion the momentum $vm = \hbar k = (2E/m)^{1/2}$ is constant, the action $S_p(k)$ is given by $\hbar k L_p$, where L_p is the length of the cycle p, and the quantum amplitude (3) associated with the cycle p is simply $t_p = |\Lambda_p|^{-1/2}\exp(ikL_p + in_p\pi)$. Here n_p is the number of bounces, and comes from the phase loss at every reflection (the boundary condition is $\psi|_{\text{disk}} = 0$).

We have chosen here the three-disk scattering system because it captures the essential topology, stability, and phase-space structure of cycles in 2D nonintegrable potentials without the complication typical of motions in generic smooth potentials (pruning of the symbolic dynamics, intermittency effects due to marginally stable orbits; we shall return to these elsewhere). The dynamics here is geometrical optics, so the cycles are faster to compute than for arbitrary smooth potentials, and we were aided much by the work of Gaspard and Rice[9] in checking our results.

The prerequisite for efficient use of curvature expansions is firm control of the symbolic dynamics.[7,13] For sufficiently separated disks, the symbolic dynamics is a ternary dynamics with alphabet {1,2,3} (the label of the disk the pinball bounces off) and a single pruning rule prohibiting consecutive repeats of the same symbol.[9,11] The corresponding curvature expansion (1) is straightforward, and converges well.[13] However, the C_{3v} point group invariance[14] of the three-disk problem simplifies and improves the curvature expansions in a rather beautiful way, which we now briefly sketch. The prime cycles fall into three classes of distinct symmetry; those invariant under rotations by $2\pi/3$ (multiplicity 2), those invariant under reflections on symmetry axes (multiplicity 3), and the rest (multiplicity 6). By use of the standard character tables[14] it can be shown[13] that the corresponding contributions to the Euler product (1) factorize as follows:

$$(1-t_p)^2 = (1-t_p^{1/3})(1-t_p^{1/3})(1+t_p^{1/3}+t_p^{2/3})^2,$$
$$(1-t_p)^3 = (1-t_p^{1/2})(1+t_p^{1/2})[1-(t_p^{1/2})^2]^2, \quad (4)$$
$$(1-t_p)^6 = (1-t_p)(1-t_p)(1-t_p)^4.$$

The three factors in this product contribute to the C_{3v} irreducible subclasses A_1, A_2, and E, respectively, and the three-disk ζ function factorizes into $\zeta = \zeta_+ \zeta_- \zeta_E^2$. Because of the symmetry, any three-disk cycle can be pieced together from segments passing through the *fundamental domain* (see Fig. 1). The $t_p^{1/2}$ and $t_p^{1/3}$ weights in (4) have direct physical meaning: They are the weights of the corresponding cycles restricted to the fundamental domain.

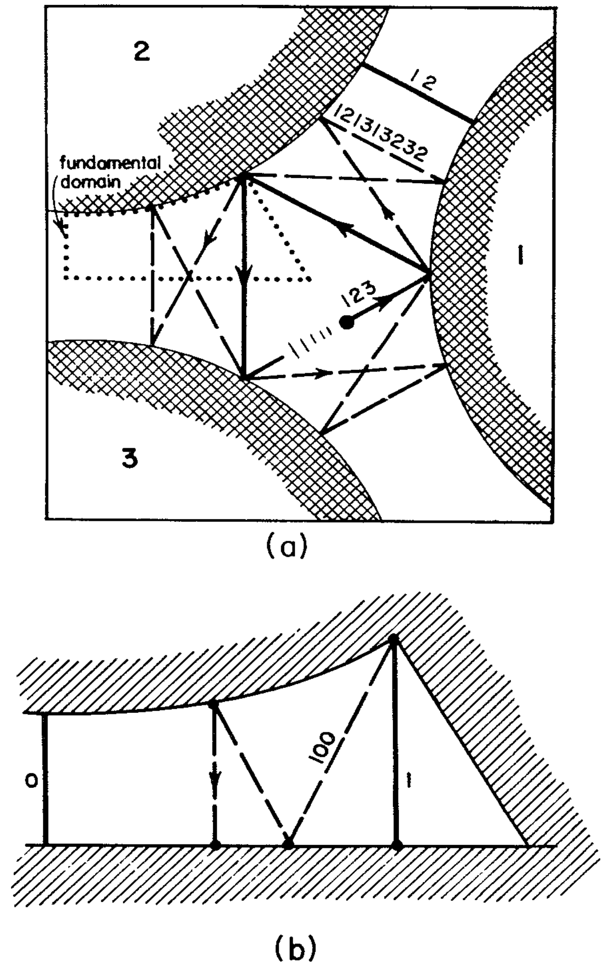

FIG. 1. The scattering geometry for the disk radius to separation ratio $a:R = 1:2.5$. (a) The three disks, with $\overline{12}$, $\overline{123}$, and $\overline{121232313}$ cycles indicated. (b) The fundamental domain, i.e., a wedge consisting of a section of a disk, two segments of symmetry axes acting as straight mirror walls, and an escape gap. The above cycles restricted to the fundamental domain are now the two fix points $\overline{0}$ and $\overline{1}$ and the $\overline{100}$ cycle.

Restriction to the fundamental domain also simplifies the symbolic dynamics: It becomes binary, with no restrictions on allowed sequences.[11,13,15] The ternary three-disk {1,2,3} labels are converted into the binary fundamental domain labels {0,1} by marking the backscatter by 0 and scatter to the third disk by 1. For example, $\overline{23} = \ldots 232323 \ldots$ maps into $\ldots 000 \ldots = \overline{0}$ (and so do $\overline{12}$, $\overline{13}$), $\overline{123} = \ldots 12312 \ldots$ maps into $\ldots 111 \ldots = \overline{1}$, and so forth (see Fig. 1).

The Euler product (1) on each irreducible subspace is easily evaluated using the factorization (4). On the symmetric A_1 and the antisymmetric A_2 subspaces, the ζ_+ and ζ_- are given by the standard curvature expansion for the binary dynamics:[1,7]

$$1/\zeta_{\mp}=(1\pm t_0)(1-t_1)(1\pm t_{10})(1-t_{100})(1\pm t_{101})(1\pm t_{1000})\cdots$$
$$=1\pm t_0-t_1\pm(t_{10}-t_1t_0)-(t_{100}-t_{10}t_0)\pm(t_{101}-t_{10}t_1)-(t_{1001}-t_1t_{001}-t_{101}t_0+t_{10}t_0t_1)-\cdots\,, \quad (5)$$

while for the mixed-symmetry subspace E the curvature expansion is given by

$$1/\zeta_E=(1+t_1+t_1^2)(1-t_0^2)(1+t_{100}+t_{100}^2)(1-t_{10}^2)\cdots$$
$$=1+t_1+(t_1^2-t_0^2)+(t_{100}-t_1t_0^2)+t_{1001}+(t_{100}-t_1t_0^2)t_1-t_{10}^2\cdots\,. \quad (6)$$

Given the curvature expansions (5) and (6), the calculation is straightforward. Following Ref. 9, we set the disk radius $a=1$, fix the disk-disk center separation $R=6$ (for the sample values listed here), compute the eigenvalues and lengths of prime cycles up to five bounces (total of fourteen cycles), substitute them into the curvature expansions, and determine the complex zeros; some hundred quantum resonances are easily determined,[13] with accuracy as good as seven significant digits for the resonances closest to the real axis. A detailed discussion of this spectrum will be presented elsewhere;[12] here we only wish to illustrate the quality of the curvature expansions. In Table I we list a few typical results, illustrate their convergence by computing them with different maximal length cycles, and compare them to the numerical solutions for poles of the exact quantum scattering matrix.[12] The convergence of the curvature expansions is striking; they are many order of magnitude more accurate than the estimates by other methods, and the gain in efficiency is dramatic: While the quantum scattering matrix requires computation of large truncations [of order of (70×70)] of infinite matrices with Bessel-functions entries, the curvature expansions require evaluation of some dozen complex exponentials and a couple of sums and differences. The judicious use of symmetry helps considerably; for example, going to the fundamental domain often *doubles* the number of significant digits for a given cycle length. The estimates could be further improved by extrapolations and knowledge of the analyticity properties of $1/\zeta$, e.g., the positions of poles.[7,13]

No poles of the infinite product can occur within the half-plane of absolute convergence.[16] This also leads to an upper bound on the resonance lifetimes. The abscissa of absolute convergence can be determined as the leading zero of (5) with t_p replaced by $|t_p|$; Im(k) for all zeros of $1/\zeta$ must lie below k_c (for the present $a{:}R=1{:}6$ example, $k_c=-0.121\,557\ldots$). The convergence of the curvature expansion is best near k_c (the longest-lived resonances) and deteriorates as one moves further in the imaginary k direction. As Rek grows, the density of resonances increases. Implications for the curvature expansion are that a certain number of terms have to be included before two resonances are distinguished; thereafter one again observes rapid convergence. This is illustrated in Table I for the resonances k_2 and k_3, which have the same $n=1$ approximation.

The explicit curvature expansions like (5) and (6) perhaps make it easier to explain our introductory claims about the exponential convergence of curvature expansions. A typical curvature expansion term involves a

TABLE I. Representative classical escape rate and quantum resonance results, illustrating the convergence of the curvature expansions. The calculations listed here are for disks with ratio radius to separation $a{:}R=1{:}6$. The first column gives the maximal cycle length used, the second the estimate of the classical escape rate from the full three-disk expansion, and the third from the fundamental domain expansion. For comparison, numerical simulation (Ref. 9) yields $\gamma=0.410\ldots$, and the $n=2,3$ approximations of Ref. 9 yield 0.3102, 0.4508, respectively. The remaining columns illustrate convergence of "typical" quantum resonances from the A_1 subspace. For comparison, the exact quantum values (Refs. 9 and 12) are given in the last row; the $n=3$ approximation of Ref. 9 gives $k_1=8.354-i0.342$.

	Escape rate		Quantum poles					
n	Full three disks	Fund. dom.	Re(k_1)	Im(k_1)	Re(k_2)	Im(k_2)	Re(k_3)	Im(k_3)
1		0.407693	8.35954	−0.32103	27.36117	−0.15486	27.36117	−0.15486
2	0.43578	0.410280	8.26784	−0.28354	27.25948	−0.16044	27.68086	−0.36148
3	0.40491	0.410336	8.27549	−0.27576	27.25883	−0.15587	27.67331	−0.33837
4	0.40945	0.410338	8.27662	−0.27696	27.25922	−0.15561	27.67015	−0.33556
5	0.41037	0.410338	8.27640	−0.27712	27.25925	−0.15562	27.66956	−0.33531
6	0.41034							
Exact			8.2611	−0.2749	27.2548	−0.1555	27.6661	−0.3340

long cycle $\{ab\}$ minus its shadowing approximation by shorter cycles $\{a\}$ and $\{b\}$:

$$t_{ab}-t_a t_b=t_{ab}(1-t_a t_b/t_{ab})$$
$$=t_{ab}[1-\exp(-\Delta\mu_{ab}/2+ik\Delta S_{ab})]\,,$$

where $\Delta\mu_{ab}=\mu_a+\mu_b-\mu_{ab}$ and $\Delta S_{ab}=S_a+S_b-S_{ab}$. The exponential falloff of curvatures is a consequence of the smallness of the term in the brackets; $\Delta\mu$ and ΔS are exponentially small for long orbits,[17] typical $O(e^{-\mu_{ab}})$. Therefore, to resolve some scale Δk, one has to keep all *difference* actions $>1/\Delta k$. Their number increases like $(\Delta k)^{h/\lambda}$, i.e., roughly linearly (if the topological entropy h equals the average Lyapunov exponent λ), and not exponentially, as one might expect naively.

To summarize, we have demonstrated that the curvature expansions are a very efficient way of evaluating the classical and quantum periodic-orbit sums. The essential ingredient for this success has been the physical insight that the dynamical ζ functions expanded this way utilize the shadowing of arbitrarily long orbits by shorter cycles; the technical prerequisite for implementing this shadowing is a good understanding of the symbolic dynamics of the classical dynamical system. Exploiting the symmetries of the problem, we are able to compute accurately a large number of resonances, using as input the actions and eigenvalues of as few as 2–14 prime cycles.

We conclude with three more general comments on the relation of classical and quantum chaotic dynamics.

(1) The curvature expansion approach presented here applies to strongly chaotic (nonintegrable) systems, and is thus a quantization scheme for a class of systems complementary to those amenable to torus quantization. (2) The symmetry factorization (4) of the dynamical ζ function is intrinsic to the classical dynamics, and not a special property of quantal spectra (in which context it was used before[9]). (3) For the strange sets studied in Refs. 1 and 7, the curvature expansion is believed to be the *exact* perturbation expansion for classical chaotic averages: Even though each cycle carries with it only its linearized neighborhood (analog of the stationary-phase approximation in the derivation of the quantum Gutzwiller sum), the union of the periodic points captures the invariant content of the full underlying smoothly curved dynamics. We expect similarly the quantum curvature expansion to overcome the limitations due to the linearization around the individual trajectories.

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