INFRA-RED STRUCTURE OF YANG-MILLS THEORIES

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An analysis of QCD magnetic moment shows that all infra-red divergences are contained in the coupling constant renormalization. They are controlled by the renormalization function β for the pure Yang-Mills field.

A general argument shows that the unrenormalized amplitude for the magnetic moment is infra-red (IR) finite in both QED and QCD (Quantum chromodynamics), but that in QCD the coupling constant renormalization introduces IR divergences. I exhibit them by computing the renormalization on the mass-shell, and show that they are controlled by the same renormalization group function β that characterizes the ultra-violet (UV) behaviour of the pure Yang-Mills field. If the renormalization is computed off-mass-shell, the IR poles in $1/\epsilon$ get replaced by the powers of $\ln\mu$ (dimension = $4 - \epsilon$, renormalization point = μ). In the off-mass-shell renormalized QCD the infra-red divergences manifest themselves through logarithmic corrections to the long range potential, leaving open a possibility of a (non-perturbative) confinement.

QCD considered here is a Yang-Mills theory of equal mass ($m \neq 0$) quarks of n colours and N strictly massless gluons. I shall consider gluonic corrections to the electromagnetic magnetic moment of a quark. The quark carries both electric charge and colour, and scatters in a weak external magnetic field. The gluons carry only colour. The magnetic moment anomaly is defined as [1]

$$a = \frac{M(\alpha_0)}{1 + L(\alpha_0)} = \sum_{n=1}^{\infty} a_0^{(n)} (\alpha_0/\pi)^n \qquad (1)$$

where $M = F_2(0)$, $1 + L = F_1(0)$ are computed from the unrenormalized proper vertex $\Gamma^\nu(p, q) = F_1(q^2)\gamma^\nu + F_2(q^2)\mathrm{i}((\sigma^{\nu\mu}q_\mu)/2m)$ given by the sum of all one-particle irreducible quark-quark-photon vertex diagrams with internal gluons, Fadeev-Popov ghosts, quark loops and quark mass counterterms. The UV finite expression for the anomaly (1) is obtained by the charge renormalization

$$\alpha = Z\alpha_0, \qquad Z^{1/2} = \frac{Z_2}{Z_1} Z_3^{1/2} = \frac{Z_2^{\mathrm{F}}}{Z_1^{\mathrm{F}}} Z_3^{1/2} = \frac{Z_3^{3/2}}{Z_{\mathrm{YM}}} \qquad (2, 3)$$

where Z's are the usual renormalization constants, computed as a power series in α_0 (subscript YM refers to 3-gluon vertex, and superscript F to the Fadeev-Popov ghost). The Z's are related by the Taylor-Slavnov [2, 3] identities (3).

The anomaly (1), expressed in terms of the unrenormalized coupling constant, is free of infra-red divergences for both QED and QCD (i.e., all $a_0^{(n)}$ are IR finite). This is particularly easy to demonstrate using the method for separation of IR divergences given in ref. [4]. The method applies to both QED and QCD, provided that for the gluon subdiagrams the gauge invariant subsets are taken together. The analysis is rather technical, but the result is very intuitive: a Feynman integral M_{G} contributing to the magnetic moment is IR divergent whenever the corresponding diagram G can be split into a vertex subdiagram S and a cloud of soft gluons attached to the external quark lines (diagram G/S), as in fig. 1a. In general some gluons in G/S can be hard, provided they are within UV divergent subdiagrams, such as, for example, the vertex subdiagram {1, 2, 3} in fig. 1a. Such subdiagrams can even contain quark loops. In all cases, the IR divergent part of M_{G} factors into M_{S} times the IR divergent part of $L_{\mathrm{G/S}}$, where $L_{\mathrm{G/S}}$ is the charge form factor computed from the diagram G/S.

One can attempt to define the IR finite part of M_{G} as $M_{\mathrm{G}} - M_{\mathrm{S}}L_{\mathrm{G/S}}$, or, for the whole perturbation

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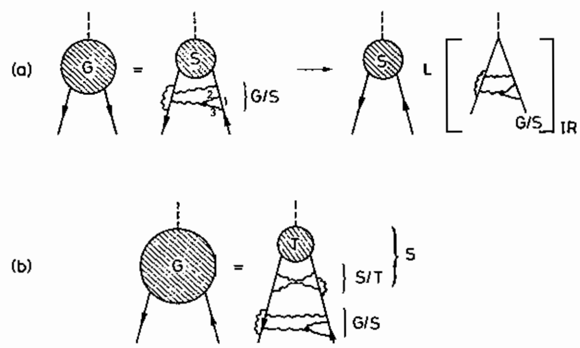

Fig. 1. a) A magnetic moment Feynman diagram G which can be split into an internal magnetic moment diagram S, and a cloud of soft gluons, attached to external (mass-shell) quarks. The IR divergences factor as $M_S(L_{G/S})_{IR}$. b) A case where the subdiagram S can be split further into T and S/T. All gluons in G/S are exterior with respect to the gluons in S/T.

series, as $M - ML$. This removes all IR divergences of the type illustrated in fig. 1a, but introduces new IR divergences not present in M. The reason is that M_S might also be IR divergent, so that the counterterm contains an IR divergence of type $M_T L_{S/T} L_{G/S}$ (see fig. 1b for the notation). Such a divergence in which an internal gluon cloud S/T is uncorrelated with the exterior gluons of G/S is absent from M_G; interior gluons can contribute to IR only if the exterior gluons are already soft. Hence the counterterm $-M_S L_{G/S}$ has to be replaced by $-(M_S - M_T L_{S/T})L_{G/S}$, or, for the whole series, $-ML$ by $-(M - ML)L$. Clearly the new counterterm $+ML^2$ will itself have to be replaced by $(M - ML)L^2$, and so forth, yielding

$$(\text{IR} - \text{finite part of } M) = M - ML + ML^2 - ML^3 + \ldots$$

$$= a.$$

This completes the demonstration of the IR finiteness of the anomaly (1), expressed in terms of the unrenormalized coupling. This has been checked explicitly up to two loop level by Korthals Altes and De Rafael [5] and by the present author [6]. As coefficients $a_0^{(n)}$ in (1) are IR finite, *all IR divergences are contained in the coupling constant renormalization Z.*

I believe that the above result is a special case of Kinoshita's [7] theorem, and that for more general QCD scattering processes the cross-sections expressed in terms of bare coupling [and summed over appropriate degenerate states‡] are also IR finite. This has

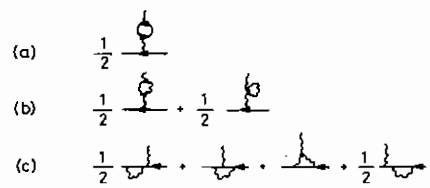

Fig. 2. A set of lowest order contributions to the coupling constant renormalization $Z^{1/2}$. Factors 1/2 come from the expansion of (3) in terms of proper self-energy and vertex graphs. The gluon loop diagrams include the Fadeev-Popov ghosts and the combinatoric factors. Z can also be computed from gluon or ghost renormalizations.

been checked to low orders by many authors [8–12] If this is true, then it is not necessary to study various QCD processes with complicated bremsstrahlung contributions; all QCD IR divergences are contained in the coupling constant renormalization Z.

It is therefore important to examine Z in more detail. One loop contributions (from (3)) are given in fig. 2. Each of the integrals can be split into an UV divergent part, independent of masses and external momenta, and an UV convergent remainder. In the diagrams with internal quark lines this occurs automatically by the rules of ref. [4]. For purely gluonic integrals (fig. 2b) a mass-scale which separates UV and IR has to be introduced externally. The UV parts of fig. 2 (and in general, Z to any order) can be absorbed into integrals contributing to $a_0^{(n)}$ in (1), where they provide a pointwise cancellation [4] of all UV singularities. Let us furthermore note that the UV part of Z is gauge independent and calculable to all orders in terms of the renormalization group function $\beta(\alpha)$ [13–16]

$$Z_{UV} = \exp\left(\int_0^\alpha \frac{dx}{x} \frac{\beta(x)}{\beta(x) - \epsilon/2} \right). \quad (4)$$

(For QCD β is known up to two loop level [17, 18].)

Having thus absorbed the UV parts of the renormalization Z, I evaluate the UV convergent remainder on the mass-shell [19]. Diagrams of fig. 2b, c develop IR divergences. In QED they cancel, but in QCD they add up to a gauge invariant IR pole

‡ Colour summations have to include all quark-soft gluon combinations such that the over-all colour of an external quark and its gluon cloud is fixed.

$$(Z^{1/2}-1)_{\mathrm{IR}} = -\tfrac{11}{3} C_{\mathrm{A}} \frac{\alpha}{4\pi} \frac{1}{\epsilon}. \tag{5}$$

Alternatively one can evaluate the UV convergent remainder off mass-shell, at external gluon momentum $-q^2 = \mu^2$. This merely replaces $1/\epsilon$ by $\ln \mu$, giving the result noted independently by other authors [50, 10, 20] in different QCD calculations.

Korthals Altes and De Rafael [5] have pointed out that the coefficient of $1/\epsilon$ is the lowest order term in the β function for the pure Yang-Mills field. Why does the function that controls the UV behaviour also control the IR behaviour? This is easiest to see when Z is evaluated on the mass-shell. There the contributions of fig. 2b, c vanish separately by Ward identity [19, 21]. This occurs through the cancellations between the IR and UV divergences. UV divergences are given by the β function through (4), hence *the IR divergences are given by the β function for the pure Yang-Mills field* as well. Now it is clear that the QCD infra-red will not be given by any simple exponentiation; knowledge of the entire β function is necessary. Even the leading singularities do not exponentiate, as one can easily check by keeping only the lowest order contribution to β in (4).

If the mass-shell coupling is finite, we see that any process computed in QCD is divergent order by order in perturbation theory. This is not surprising: the renormalization group tells us that QCD is asymptotically free, but that the effective coupling diverges as we approach the mass-shell. Conversely, if the mass-shell coupling is finite, asymptotic freedom starts infinitesimally close to the mass-shell. Such theory is not just asymptotically free, but free everywhere.

However, just as one absorbs the UV divergences into the bare coupling, one is free to absorb the IR divergences into the mass-shell coupling. In that case it is necessary to give a prescription for computing the finite parts of the renormalization, that is, define the mass scale which separates UV and IR regions for the pure Yang-Mills field. Ideally, such scale will be introduced by quark binding. At this time, it is standard to assume a finite effective coupling at some $-q^2 = \mu^2$. In this case all QCD processes (properly summed over degenerate states) are IR finite [8, 9, 10, 12] in the sense that $\ln \mu$ is finite, and dressed coloured particles can appear in the asymptotic states. Now the IR divergences manifest themselves as $\ln^n(-q^2)$ modifications to the large distance ($q \to 0$) colour potential. Whether they, summed to all orders, change the potential sufficiently to produce confinement, is still an open question. The connection of QCD IR singularities to the renormalization group, illustrated here, might lead to the answer.

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