Project Summary

A dynamical theory of turbulence is one of *the* grand outstanding challenges of classical physics: how are we to describe unstable dynamics of very many degrees of freedom? The **recurrent patterns program** proposed here is an undertaking very different from most current turbulence research. Here everything follows from deterministic dynamics, with no probabilistic velocity distribution, external stochastic forcing, or homogeneity of turbulence assumptions. The theory emphatically does *not* seek to be universal; its goal is to accurately predict measurable effects of a turbulent flow, such as the frictional drag, for a given system with given physical parameters and boundary conditions. Inspired by the unstable coherent structures observed in turbulence, the proposed theory follows a path also radically different from the long-time PDE simulation approaches. It postulates, in a nutshell, that (mesh size$\times$ time)$^{\#\text{ fields}} \approx 10^{10}$ is a vastly too large a number, and that a finite set of *building blocks*, of order of $10^2 - 10^4$ recurrent patterns, is *all that is required for the eventual assembly of a predictive capability for non-equilibrium turbulent flows*.

The key issues are (1) *how* to identify, and (2) *what* to do with these patterns? In the initial phase, a new method for determining recurrent patterns will be applied to a turbulent 1-*d* model system. With theoretical refinements, *turbulent regimes of full 3-d hydrodynamic flows should be computationally within reach, and will be investigated.* An implementation of the Sinai-Ruelle-Bowen ergodic far-from-equilibrium natural measure, the recurrence frequency with which a turbulent flow visits a given state space region, is a great challenge in the case of infinite-dimensional recurrent patterns. We propose to build up this measure from small, computable recurrent patches, assembled using the PI's *cycle expansions* theory.

Intellectual Merit: Turbulence is *the* unsolved problem of classical physics; a big conceptual gap separates what the dynamical systems approaches have achieved so far for few degrees-of-freedom chaotic systems, and what needs to be done as systems grow large and turbulent. A distinctive aspect of the proposed research is its integration of diverse areas of expertise: The problem of *how* to extract the recurrent patterns is an intensely numerical undertaking, a domain of fluid dynamicists. The theory for *what* to do with this infinity of patterns is the domain of ergodic theorists. We bring to the project both sets of skills. *This proposal addresses the grand challenge of nonlinear science: Describe theoretically and explore numerically and experimentally the dynamics of high-dimensional nonlinear systems. Furthermore, this proposal will foster applications of the methods thus developed to engineering problems, such as turbulent drag reduction.*

Broader Impact: The PI intends to assemble a strong US, European and Japanese collaborative team from natural sciences (physics, atmospheric sciences) and engineering (aerospace, fluid dynamics), with complementary experimental, numerical and theoretical skills. The proposed research will be carried out in part globally, within the context of the very successful electronic `ChaosBook.org` webbook collaboration, and in part locally, at the Georgia Institute of Technology *Center for Nonlinear Science*, a uniquely interdisciplinary environment in which training of undergraduates, graduate students, and postdocs in mathematical methodology of complex systems is driven by concrete physical problems. The record of bringing students into front-line research is excellent: the "recurrent patterns program" itself initially arose from a student contribution to `ChaosBook.org/projects`.

Project description:

Turbulence: A walk through a repertoire of unstable recurrent patterns

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Pushing the current limits of what is numerically attainable in solving large sets of PDEs, the best we currently can do is to write down a set of coupled hydrodynamical/chemical PDEs, discretize it on a mesh or a basis function set of order of 10^4-10^9 components, with velocity and concentration fields at each point, and follow numerically trajectories in high-dimensional phase spaces. Due to the sensitive dependence on initial conditions, and dramatic qualitative differences in dynamics at different model parameter values, already at moderate Rayleigh numbers the available nonlinear dynamics tools largely fail.

Nevertheless, insights gleaned from nonlinear dynamics have greatly changed the ways we think about turbulence. The field theories that describe the classical world — the strongly nonlinear hydrodynamics, magneto-hydrodynamics, complex Ginzburg-Landau equations — are turbulent and have bewildering wealth of solutions, with very few of the important ones analytical in form.

A systematic exploration of solutions of such systems has so far been implemented only for one of the very simplest field theories, the 1-dimensional Kuramoto-Sivashinsky system, within what we shall here refer to as the "Hopf's vision" or "recurrent patterns program". In its spirit, this program is very different from most ideas that animate current turbulence research. It is distinct from the Landau quasi-periodic picture of turbulence as a sum of infinite number of incommensurate frequencies. It is not the Kolmogorov's 1941 homogeneous turbulence, with no coherent structures fixing the length scale, here all the action is in specific coherent structures. It is emphatically *not* universal; spatiotemporally periodic solutions are specific to the particular set of equations and boundary conditions. And it is *not* probabilistic; everything is fixed by the deterministic dynamics, with no probabilistic assumptions on the velocity distributions or external stochastic forcing.

The theory of deterministic chaos tells us that the starting approximation to strongly nonlinear systems should be quite different from the traditional point of departure, the nearly periodic motions. Portraits of chaotic systems exhibit amazingly rich self-similar structure which is not at all apparent in their formulation in terms of differential equations. The theory proposed here follows a path radically different from direct long-time PDE simulations. It postulates, in a nutshell, that $(\text{mesh size} \times \text{time})^{\#\ \text{fields}} \approx 10^{10}$ is a vastly too large a number, and that a finite set of *building blocks*, of order of $10^2 - 10^4$ recurrent patterns, is all that is *required for the eventual assembly of a predictive capability for non-equilibrium turbulent flows.*

A method for numerical determination of recurrent patterns is described in sect. 3. In sect. 4 we introduce the Kuramoto-Sivashinsky system, and in sect. 5 we review the status of determining such patterns for full three-dimensional Navier-Stokes turbulence. The theory of computing averages in spatio-temporally chaotic dynamics in terms of spatio-temporally recurrent unstable patterns is reviewed in sect. 6, and the detailed plan of the proposed research is outlined in sect. 7.

1 Background

The program that we propose to follow here was lucidly outlined by Eberhardt Hopf in a beautiful 1948 paper [1]; his prose is as good as anything written since on the subject, so we introduce the program in Hopf's own words:

"Consider an incompressible and homogeneous viscous fluid within given material boundaries under given exterior forces. The boundary conditions and the outside forces are assumed to be stationary, i.e. independent of time. For that, it is not necessary that walls be at rest themselves. Part of the material walls may move in a stationary movement, provided that the geometrical boundary as a whole stays at rest. An instance is [$\cdots$] a fluid between two parallel planes which are translated within themselves with given constant velocities."

In sect. 5 we review the current status of this problem, "the constrained plane Couette turbulence", and explain why we believe that today Hopf's program can be implemented.

E. Hopf continues: *"It is convenient to visualize the solutions in the phase space Ω of the problem. A phase or state of the fluid is a vector field $u(x)$ in the fluid space that satisfies [the Navier-Stokes equations] and the boundary conditions. The totality Ω of these phases is therefore a functional space with infinitely many dimensions. A flow of the fluid represents a point motion in Ω and the totality of these phase motions forms a stationary flow in the phase space Ω, which, of course, is to be distinguished from the fluid flow itself. What is the asymptotic future behavior of the solutions, how does the phase flow behave for $t \to \infty$? And how does this behavior change as [the viscosity] μ decreases more and more? How do the solutions which represent the observed turbulent motions fit into the phase picture? The great mathematical difficulties of these important problems are well known and at present the way to a successful attack on them seems hopelessly barred. There is no doubt, however, that many characteristic features of the hydrodynamical phase flow occur in a much larger class of similar problems governed by non-linear space-time systems. In order to gain insight into the nature of hydrodynamical phase flows we are, at present, forced to find and to treat simplified examples within that class."*

One of the simplest and today extensively studied spatially extended dynamical systems, reviewed here in sect. 4, is the 1-spatial dimension Kuramoto-Sivashinsky [2] system that describes the flutter of a gas flame. For this relatively simple system, we have checked Hopf's vision in detail [3]: many recurrent patterns have been determined numerically, and the recurrent-patterns theory predictions tested for various values of flame front parameters. But, as we shall argue in sect. 5, with modern computation and new ideas, the way to a successful attack on the full Navier-Stokes problem is no longer "hopelessly barred".

E. Hopf: *"The observational facts about hydrodynamic flow reduced to the case of fixed side conditions and with μ as the only variable parameter are essentially these: For μ sufficiently large, $\mu > \mu_0$, the only flow observed in the long run is a stationary one (laminar flow). This flow is stable against arbitrary initial disturbances. Theoretically, the corresponding exact solution is known to exist for every value of $\mu > 0$ and its stability in the large can be rigorously proved, though only for sufficiently large values of μ. The corresponding phase flow in phase space Ω thus possesses an extremely simple picture. The laminar solution represents a single point in Ω invariant under the phase flow. For $\mu > \mu_0$, every phase motion tends, as $t \to \infty$, toward this laminar point. For sufficiently small values of μ, however, the laminar solution is never observed. The turbulent flow observed instead displays a complicated pattern of apparently irregularly moving "eddies" of varying sizes. The view widely held at present is that, for $\mu > 0$ having a fixed value, there is a "smallest size" of eddies present in the fluid depending on μ and tending to zero as $\mu \to 0$. Thus, macroscopically, the flow has the appearance of an intricate chance movement whereas, if observed with sufficient magnifying power, the regularity of the flow would be never doubted.*

The qualitative mathematical picture which [E. Hopf] conjectures to correspond to the known facts about hydrodynamic flow is this: To the flows observed in the long run after the influence of the initial conditions has died down there correspond certain solutions of the Navier-Stokes equations.

These solutions constitute a certain manifold $\mathcal{M} = \mathcal{M}(\mu)$ invariant under phase flow. Probably owning to viscosity $\mathcal{M}$ has a finite number $N = N(\mu)$ of dimensions. This effect of viscosity is most evident in the simplest case of μ sufficiently large. In this case $\mathcal{M}$ is simply a point, $N = 0$. Also the complete stability of $\mathcal{M}$ is in this simplest case is obviously due to viscosity. On the other hand, for smaller and smaller values of μ, the increasing chance character of the observed flow suggests that $N \to \infty$ monotonically as $\mu \to 0$."

Today $\mathcal{M}(\mu)$ is known as the "inertial manifold" and widely studied. Its finite dimensionality for non-vanishing μ has been, in certain settings, rigorously established by Foias and collaborators [4]. Hopf's "monotonicity" with system parameters such as μ is a dream - a family of dynamical systems can do anything, including collapsing to a 1-dimensional periodic attractor, infinitely often [5]. Nevertheless, the intuition is qualitatively correct. Inspired by his study of the first (Hopf) bifurcation from stationary to oscillatory behavior, E. Hopf proposes in the 1948 paper what is today known as the Landau-Hopf model of turbulence [1, 6], with turbulence arising from an infinite sequence of Hopf bifurcations of incommensurate frequencies, and the dynamics taking place on a large-dimensional torus. In hindsight, this is not a good model. In a fundamental 1971 paper Ruelle and Takens [7] have shown that nonlinearities lead to turbulent dynamics confined to a strange attractor $\mathcal{M}(\mu)$ of fractal dimension. Our claim is that the hierarchy of recurrent patterns of longer and longer periods cover this fractal set in a systematic fashion.

What E. Hopf writes next is prescient: *"The geometrical picture of the phase flow is, however, not the most important problem of the theory of turbulence. Of greater importance is the determination of the probability distributions associated with the phase flow [$\cdots$]."* This is the key challenge, the one in which the theory of dynamical systems has made the greatest progress in the last half century: the Sinai-Ruelle-Bowen ergodic theory of "natural" or SRB measures for far-from-equilibrium systems. This is the challenge the proposal addresses employing the methodology sketched in sect. 6, and developed in depth in `ChaosBook.org`, our electronic textbook. [8]

2 Recurrent patterns: a repertoire of turbulence?

Spatially extended systems are systems with very many coupled degrees of freedom whose dynamics range from ordered to very disordered and turbulent. Theoretical challenges in learning to control such systems require understanding the instability mechanisms arising from the interplay between nonlinearity, transient dynamics, and stochasticity, often resulting in rather surprising dynamics. For instance, the onset of turbulence observed in shear flows occurs well within the Reynolds number regime where the laminar flow is still linearly stable. Transient amplification in such systems leads to extreme sensitivity to noise, precluding any spatially localized control, even at physically unrealistic low levels of noise [10].

Figure 1: *Planktonic copepods accelerate their swimming appendages up to $50 m/s^2$. Two vortices are created by 4 pairs of legs galloping twice within the field of view [9].*

The inspiration for the "recurrent patterns program" comes from the way we perceive turbulence. Turbulent systems never settle down, but we can identify a snapshot as a "cloud", and an experimentalist can tell what the values of physical parameters in a turbulence experiment were after a glance at the digitized image of its output. Still more intriguingly, even in a turbulent sea, a plankton can identify a prey, predator, or mate by the complex but characteristic wakes that bodies moving through fluid shed, figure 1. How do we do it?

An answer was offered by E. Hopf [11]. In Hopf's vision turbulence explores a repertoire of distinguishable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar whorl, figure 2.

At any instant and a given finite spatial resolution the system approximately tracks for a finite time a pattern belonging to a finite alphabet of admissible patterns, and the dynamics can be thought of as a walk through the space of such patterns, just as chaotic dynamics with a low dimensional attractor can be thought of as a succession of nearly periodic (but unstable) motions. There are two issues here: (1) *how* to identify the alphabet of patterns? (2) *what* to do with this repertoire of patterns?

3 Recurrent patterns: how to find them?

The issue of *"how"* is an intensely numerical undertaking. Consider what is at stake: We need to compute, or extract from data, many distinct 3-dimensional color videos of patches of turbulence. Just in terms of memory we need something of order of a gigabyte to store a single pattern. If a given image is an instant in the motion of a turbulent swirl, nature computes the next instant by letting every particle jostle its neighbors. We do it by time consuming numerical pixel-by-pixel calculations. That is still not so hard, but the theory proposed here seeks a 3-dimensional pattern that repeats itself *exactly* after a (yet to be determined) time period. *A priori*, we have no clue what such patterns look like, how big and what shapes are the basic spatial units, or what their time periods should be. So in our initial guess image *all* pixels are wrong, and we need to keep jostling them numerically until gigabytes of pixels settle into a pattern allowed by the laws of motion. Not only that, but a computer needs variable resolution, as small regions can play a very important role. All that takes computer time, lots of it. The theory then requires that we compute many such patterns, and be sure that we have found *all* of the patterns up to given spatial extent and time period within the desired tolerance limits.

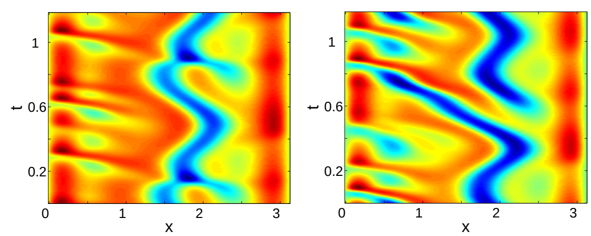

Figure 2: *A pattern and its near recurrence in a Kuramoto-Sivashinsky simulation: the swirl on the left recurs approximately within another pattern, right image. The color/greyscale codes the height of the flame front. [14]*

Clearly, conceptually new methods applicable to high numbers of deterministic degrees of freedom as well as to mixed systems of chaotic, integrable, and stochastic components need to be developed; the present dearth of such methods remains a fundamental barrier to the direct application of concepts of nonlinear science to turbulent phenomena.

3.1 Recurrent pattern searches

We make here a distinction between a *periodic orbit* — a low-dimensional trajectory that folds into itself after a time T, and a *recurrent pattern* — a trajectory in the infinite-dimensional state space $\mathcal{M}$ which tiles both space and time periodically. A periodic orbit is a solution (x, T) of the *periodic orbit condition* $f^T(x) = x$ for a given flow $x \to f^t(x)$. The task we face is to determine a set of recurrent patterns of shortest periods. Any numerical solution of a PDE is based on its representation in terms of a truncated but large set of coupled nonlinear ODEs. So, in practice we always search for *periodic orbits* of flows defined by first order ODEs

$$\frac{dx}{dt} = v(x)\,, \qquad x \in \mathcal{M} \subset \mathbb{R}^d \tag{1}$$

in many (even infinitely many) dimensions d, with the vector field $v(x)$ a smooth, differentiable field. Nevertheless, we insist on referring to the problem at hand as a "recurrent pattern search", as attempts to visualize a solution in a 15,000-dimensional arbitrarily discretized spaces are a fruitless undertaking. The solutions can be understood only as fields embedded in the physical space and time.

A variety of methods for determining all unstable periodic orbits up to a given length have been devised and successfully implemented for low-dimensional systems [8]. For turbulent flows, with high (or infinite) dimensional phase spaces and complicated dynamical behavior, most of the existing methods become unfeasible in practice. The bottleneck for the recurrent patterns program has been the lack of methods for finding even the simplest recurrent patterns, and the lack of intuition as to what such patterns would look like.

PI's group has formulated and explored numerically a novel variational principle for determining unstable spatio-temporally periodic solutions of extended systems [13, 14]. The idea of the method is to make a rough but informed guess of what the desired pattern looks like globally, and then use a variational method to drive the initial guess toward the exact solution. For robustness, we replace the guess of a single pattern at a given instant by a guess of an entire orbit. For numerical safety, we replace the Newton-Raphson iteration by the "Newton descent", a differential flow that strictly minimizes a cost function computed from the deviation of the approximate flow from the true flow.

We start by guessing a *loop*, a smooth, differentiable closed curve $\tilde{x}(s) \in L \subset \mathcal{M}$, parametrized by $s \in [0, 2\pi]$ with $\tilde{x}(s) = \tilde{x}(s + 2\pi)$, and the loop tangent vector

$$\tilde{v}(\tilde{x}) = \frac{d\tilde{x}}{ds}\,, \quad \tilde{x} = \tilde{x}(s) \in L\,.$$

The initial loop is *not* a solution of the flow equations — it is at best a rough guess as to what a solution might look like.

While there is no extremal principle associated with the general flow (1), at least three separate lines of argument (cost minimization [13], over-damped Newton method [14], Wiener-Onsager-Machlup stochastic path extremization [15]) all lead to the same variational principle, minimization of the simplest cost function

$$F^2[\tilde{x}] = \frac{1}{|L|} \oint_L ds\, (\tilde{v} - v)^2\,, \quad \tilde{v} = \tilde{v}(\tilde{x}(s, \tau))\,, \;\; v = v(\tilde{x}(s, \tau))\,, \tag{2}$$

which penalizes misorientation of the local loop tangent vector $\tilde{v}(\tilde{x})$ relative to the dynamical velocity field $v(\tilde{x})$ of (1), see figure 3 (d).

Vary the cost functional $F^2[\tilde{x}]$ with respect to the (yet undetermined) fictitious time τ,

$$\frac{dF^2}{d\tau} = \frac{2}{|L|} \oint_L ds\, (\tilde{v} - v)\, \frac{d}{d\tau}(\tilde{v} - v)\,.$$

The simplest, exponentially decreasing cost functional is obtained by taking the $\tilde{x}(s, \tau)$ dependence on τ to be point-wise proportional to the deviation of the two vector fields

$$\frac{d}{d\tau}(\tilde{v} - v) = -(\tilde{v} - v)\,, \tag{3}$$

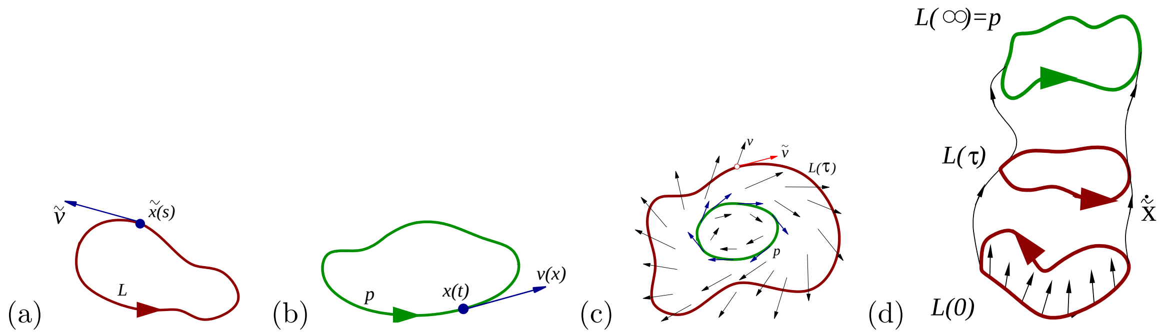

Figure 3: *(a) A loop L* defines *its tangent velocity vector $\tilde{v}$. (b) A periodic orbit p is* defined *by the vector field $v(x)$. (c) In general the orientation of the loop tangent $\tilde{v}(\tilde{x})$ does not coincide with the orientation of the velocity field $v(\tilde{x})$; for a periodic orbit p it does so at every $x \in p$. (d) An annulus $L(\tau)$ with the fictitious time flow $\tilde{x}$ connecting smoothly the initial loop $L(0)$ to a periodic orbit p.*

so the fictitious time flow drives the loop to $L(\infty) = p$ monotonically, at exponential rate figure 3 (d):

$$\tilde{v} - v = e^{-\tau}(\tilde{v} - v)|_{\tau=0} . \tag{4}$$

Making the $\tilde{x}$ dependence in (3) explicit we obtain the *Newton descent equation*, a PDE which evolves the initial loop $L(0)$ into the desired periodic orbit p in the fictitious time $\tau \to \infty$:

$$\frac{\partial^2 \tilde{x}}{\partial s \partial \tau} - A \frac{\partial \tilde{x}}{\partial \tau} = v - \tilde{v} , \qquad A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} . \tag{5}$$

Numerically, we distribute many points along a smooth loop L (successive snapshots of the the pattern at successive time instants). The infinitesimal time step and loop deformation limit corresponds to this partial differential equation. It is important to note that we do not need to solve this equation accurately along the way — only the periodic orbit itself needs to be computed with a high precision.

We start by long-time numerical runs of the dynamics, in order to get a feeling for frequently visited regions of the phase space, *i.e.* the natural measure, and to search for close recurrences [16]. An initial loop guess $L(0)$ is crafted by taking a nearly recurring segment of the orbit, smoothed and made periodic by a FFT into the wavenumber representation, dropping the high frequency components, and an FFT back to the phase space.

3.2 Future directions

(1) Inertial manifold: In devising the Newton descent method we have made a series of restrictive choices, many of which could be profitably relaxed. In particular, the choice of a *Euclidean metric cost function* $F^2[\tilde{x}]$ has no compelling merit. For a flow like the Kuramoto-Sivashinsky, the low Fourier modes a_1, a_2, ... are clearly more important than the high ones, a_k, a_{k+1}, ... , k large. A more inspired choice would use intrinsic information about dynamics, replacing $\delta_{ij}F_iF_j$ by a metric $g_{ij}F_iF_j$ that penalizes straying away in the dangerous, unstable directions more than deviations in the tame, strongly contracting ones. *The outstanding challenge and one of the core goals of the proposal is to find out how to restrict the inversion of the loop gradients to a low-dimensional inertial manifold subspace.* Without this, there is no hope of applying the theory to hydrodynamic turbulence.

(2) Topology: As for high-dimensional flows we are usually clueless as to what the solutions should look like, currently we have no way of telling to which periodic orbit the loop space flow (5) will take our initial guess, other than to the "nearest" periodic orbit of topology "similar" to the initial loop.

We have checked that the Newton descent integration yields quickly and robustly the short unstable cycles for standard models of low-dimensional dissipative flows. A more daunting challenge is a search for spatio-temporally periodic solutions of PDEs.

4 Kuramoto-Sivashinsky system

One of the simplest and extensively studied spatially extended dynamical systems is the Kuramoto-Sivashinsky system [2] (see Holmes, Lumley and Berkooz [17] for a delightful discussion of why this system deserves study),

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \tag{6}$$

which arises as an amplitude equation for interfacial instabilities in a variety of contexts, such as flame fronts. Amplitude $u(x,t)$ has compact support, with $x \in [0, 2\pi]$ a periodic space coordinate. The $(u^2)_x$ term makes this a nonlinear system, t is the time, and ν is a "viscosity" damping parameter that irons out any sharp features. Numerical simulations demonstrate that as the viscosity decreases (or the size of the system increases), the "flame front" becomes increasingly unstable and turbulent. The task of the theory is to describe this spatio-temporal turbulence and yield quantitative predictions for its measurable consequences.

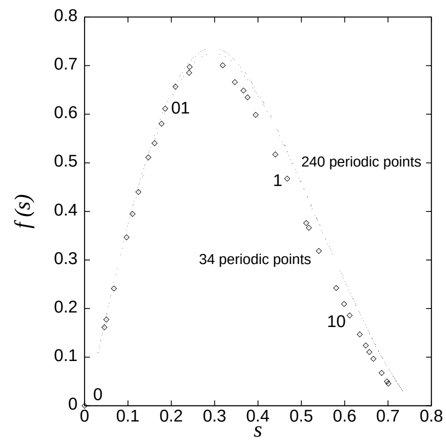

Figure 4: *The return map $s_{n+1} = f(s_n)$ constructed from 274 periodic solutions [3] of the Kuramoto-Sivashinsky equations (6), with s the distance measured along the unstable manifold of the spatio-temporally periodic solution of the shortest period.*

We have proposed that the Kuramoto-Sivashinsky system (6) be used as a laboratory for exploring the feasibility of the recurrent patterns program. We now summarize the published results obtained so far in this direction by Christiansen et al. [3] and Zoldi and Greenside [18].

The solution $u(x,t) = u(x + 2\pi, t)$ is periodic on the $x \in [0, 2\pi]$ interval, so one (but by no means only) way to solve such equations is to expand $u(x,t)$ in a discrete spatial Fourier series

$$u(x,t) = i \sum_{k=-\infty}^{+\infty} a_k(t) e^{ikx} \,. \tag{7}$$

Restrict the consideration to the subspace of odd solutions $u(x,t) = -u(-x,t)$ for which a_k are real. Substitution of (7) into (6) yields the infinite ladder of evolution equations for the Fourier coefficients a_k:

$$\dot{a}_k = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m} \,. \tag{8}$$

$u(x,t) = 0$ is a fixed point of (6), with the $k^2\nu < 1$ long wavelength modes of this fixed point linearly unstable, and the short wavelength modes stable. For $\nu > 1$, $u(x,t) = 0$ is the globally attractive stable fixed point. As one varies the burning rate, the flame front can become very unstable and turbulent: Starting with $\nu = 1$ the solutions go through a rich sequence of bifurcations, and myriad unstable periodic solutions whose number grows exponentially with period.

The essential limitation on the numerical studies undertaken so far have been computational constraints: in truncation of high modes in the expansion (8), sufficiently many have to be retained to ensure the dynamics is accurately represented. Christiansen et al. [3] have examined the dynamics for values of the damping parameter $\nu = 0.029910$, $\nu = 0.029924$, close to the onset of chaos, while Zoldi and Greenside [18] have explored somewhat more turbulent values of ν. With the recently developed variational approach of sect. 3 we can already go deeper into the turbulent regime. A typical initial loop guess is displayed in figure 5, along with the periodic orbit found by the Newton descent method. In a loop discretization (here $N = 512$ points represent the loop) each point has to be specified in all d dimensions; here the coordinates $\{a_1, a_2\}$ are picked arbitrarily. Such numerical results are only a proof of principle. We do not know to what periodic orbit the fictitious time flow will evolve for a given initial loop, other than that empirically the flow goes toward the "nearest" periodic orbit, with the largest topological resemblance.

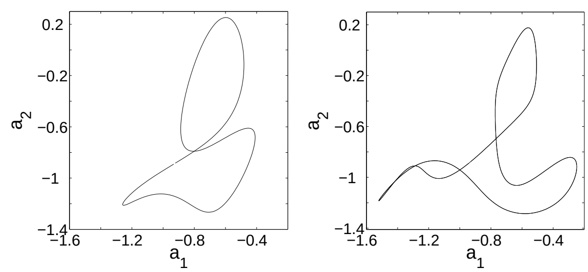

Figure 5: *Newton descent at work. Left frame: initial loop guess. Right frame: the periodic orbit p reached by the Newton descent (the left frame in figure 2).*

One pleasant surprise is that even though one is dealing with (infinite dimensional) PDEs, for strong dissipation values of parameters, the spatio-temporal chaos is sufficiently weak that the flow can be visualized as an approximately 1-dimensional Poincaré return map $s \to f(s)$ from the unstable manifold of the shortest periodic point onto its neighborhood, see figure 4. This representation makes it possible to systematically determine all nearby periodic solutions up to a given maximal period.

So far some 1,000 prime cycles have been determined numerically for various values of viscosity. The differences between various shortest period recurrent patterns are of the order of 50% of a typical variation in the amplitude of $u(x,t)$, so the chaotic dynamics is already exploring a sizable swath in the space of possible patterns even so close to the onset of spatio-temporal chaos. Other solutions, plotted in the configuration space, exhibit the same overall gross structure. Together, they embody Hopf's vision: They form the repertoire of the recurrent spatio-temporal patterns that is being explored by the turbulent dynamics.

With improvement of numerical codes considerably *more turbulent regimes should become accessible, and will be investigated within the project proposed here.*

5 Full Navier-Stokes: plane Couette turbulence

The plane shear flows are among the best studied hydrodynamic flows, both for the simplicity of the setting for fundamental studies of hydrodynamic turbulence, and for the engineering importance of the boundary layer turbulent drag. Easily visualized, they lend themselves to both colorful visualizations and colorful phenomenology, see Waleffe [19]. We believe that turbulent 3-d hydrodynamics is today within reach of the recurrent patterns program.

An example of what has already been attained, and what this project intends to push to a higher level, is the most impressive application of the recurrent patterns ideas so far, by Kawahara and Kida [20]: the first demonstration of existence of an unstable recurrent pattern in a turbulent hydrodynamic flow. They have located an important unstable spatio-temporally periodic solution in a *15,422-dimensional* numerical discretization of the three-dimensional constrained plane Couette turbulence, with no-slip boundary conditions, at $Re = 400$. A 3-d snapshot of this solution at a given instant is given in figure 6.

Figure 7 illustrates why a single unstable periodic solution might already encode essential information about the turbulent flow. In the Couette flow energy is injected locally, through the drag exerted by the turbulent fluid on the two counter-moving plane walls, and consumed globally, by the viscous dissipation at small scales. Plotted horizontally is the energy input I, and vertically the dissipation D per unit time for a long-time simulation (green), for the periodic solution figure 6 (red), and for a short nearby segment of a typical trajectory (yellow). Were the dynamics laminar, the solution would be a stationary point at $(I, D) = (1, 1)$. The turbulent solution fluctuates by almost a factor of 2, with the mean turbulence-induced drag on the "airplane wing" substantially larger than what it would have been for a laminar flow, by a factor of nearly 3.

Figure 6: *A 3-d instantaneous snapshot of an unstable periodic solution embedded in the Couette turbulence [20]. Vectors and colors code velocities and vorticities.*

Periodic orbit theory predicts the time averages and fluctuations of measurable quantities from their values computed on individual unstable periodic solutions. Accurate predictions require sets of the shortest periodic solutions. What is very encouraging about the Kawahara-Kida example is that already a *single unstable recurrent pattern* yields estimates of the mean dissipation D and mean velocity profiles across the Couette channel with a surprisingly high accuracy, presumably because it happens to explore the (infinite-dimensional) phase space region with the highest concentration of the natural measure.

6 Theory of recurrent patterns

Suppose that, armed with a computer and a great deal of skill, one has determined a set of unstable spatio-temporally periodic solutions: *What* are we to do with them? Here the periodic orbit theory comes to our rescue, with a wonderfully counterintuitive surprise: the theory [8] says that the more unstable the patterns are, the more accurate will be the predictions based on a small number of the shortest recurrent patterns! But as the short period recurrent patterns have not had sufficient time to be rendered meaningless by exponential instability, they are precisely the solutions that can be accurately (if laboriously) computed.

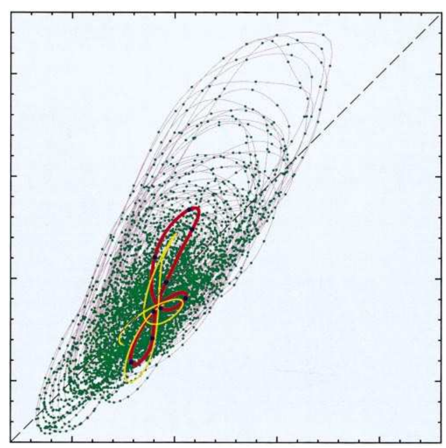

Figure 7: *Frictional drag vs. dissipation for the Couette turbulence. [20]*

Now we turn to the central issue; qualitatively, these solutions demonstrate that the recurrent patterns can be found, but how is this information to be used quantitatively? The *periodic orbit theory* [8] answers such questions by assembling individual patterns into accurate predictions for measurable global averages, such as (let us say) the diffusion of light particles by turbulent air. The key idea of the periodic orbit theory is to compute measurable averages by means of a formula which re-expresses the average as a sum over all the possible patterns grouped hierarchically, by the likelihood of pattern's occurrence.

Very briefly (for a detailed exposition consult refs. [8, 21]), the task of any theory that aspires to be a theory of chaotic, turbulent systems is to predict the value of an "observable" $a(x)$ from the spatial and time averages evaluated along dynamical trajectories $x(t)$

$$\langle a \rangle = \lim_{t\to\infty} \frac{1}{t} \left\langle A^t \right\rangle \,, \qquad A^t(x) = \int_0^t d\tau \, a(x(\tau)) \,.$$

where x(t) is a point in a high- (and in this context, infinite-) dimensional state space. The key idea of the periodic orbit theory is to extract this average from the leading eigenvalue of the evolution operator $\mathcal{L}^t(x,y) = \delta(y - x(t)) e^{\beta A^t(x)}$ via the trace formula [22]

$$\mathrm{tr}\, \mathcal{L}^t = \sum_p$$  $$_p = \sum_p \sum_{r=1}^{\infty} \frac{T_p \, \delta(t - rT_p)}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} e^{r\beta A_p} \qquad (9)$$

which relates the spectrum of the evolution operator to a sum over prime periodic solutions p of the dynamical system and their repeats r.

What does this formula mean? The intuitive meaning of a trace formula is that it expresses the average $\langle \exp(\beta A^t) \rangle$ as a discretized integral over the dynamical space partitioned topologically into a repertoire of spatio-temporal patterns, each weighted by its SRB measure, *i.e.* the likelihood of pattern's occurrence in the long time evolution of the system. More precisely: Prime cycles partition the dynamical space into neighborhoods, each cycle enclosed by a tube whose volume is the product of its length T_p and its thickness $|\det(\mathbf{1} - \mathbf{J}_p)|^{-1}$. The trace picks up a periodic orbit contribution only when the time t equals a prime period or its repeat, a constraint enforced here by $\delta(t - rT_p)$. $\mathbf{J}_p$ is the Jacobian of cycle p, so for long cycles $|\det(\mathbf{1} - \mathbf{J}_p^r)| \approx$ (product of expanding eigenvalues), and the contributions of long and very unstable cycles are exponentially small compared to the short cycles which dominate trace formulas. The number of contracting directions and the overall dimension of the dynamical space is immaterial; that is why the theory also applies to (infinite-dimensional) PDEs. All this information is purely geometric, intrinsic to the flow, coordinate re-parametrization invariant, and the same for any average one might wish to compute. The information related to a specific observable is carried by the weight $e^{\beta A_p}$, the periodic orbit estimate of the contribution of $e^{\beta A^t(x)}$ from the p-cycle neighborhood.

Periodic solutions are important because they form the skeleton of the invariant set of the long time dynamics, ordered hierarchically; short cycles give dominant contributions to the invariant set, longer cycles corrections. Errors due to neglecting long cycles can be bounded, and for nice hyperbolic systems they fall off exponentially or even super-exponentially with the cutoff cycle length [23]. Short cycles can be accurately determined and global averages (such as transport coefficients) can be computed from short cycles by means of cycle expansions.

To emphasize the obvious: the spatio-temporally periodic solutions are *not* to be thought of as eigenmodes, a good linear basis for expressing solutions of the equations of motion. Something like a dilute instant approximation makes no sense at all for strongly nonlinear systems that we are considering here. As the equations are nonlinear, the periodic solutions are in no sense additive, and their linear superpositions are not solutions.

In particular, while our approach to turbulence is motivated by the same considerations as those eloquently explained in Holmes, Lumley, and Berkooz [17], the proposed theory is very different.

Holmes *et al.* assume that steady state turbulent dynamics fills out a submanifold of phase space in an approximately Gaussian-noise way, use the proper orthogonal decomposition to determine the “center of mass” of the natural measure of the turbulent dynamics, and construct from the spatial 2-point correlation function a finite number of modes description of the turbulent dynamics restricted to the inertial manifold. The assumption that the natural measure is concentrated in an ellipsoidal neighborhood of its center of mass is natural for stochastic processes, but not so for measures concentrated on fractal sets: The “center of mass” of the Hénon attractor has little to do with the attractor, for example.

Instead, it is the trace formulas and spectral determinants of the periodic orbit theory that prescribe how the repertoire of admissible recurrent patterns is to be systematically explored, and how these solutions are to be put together in order to predict measurable observables.

7 Proposed research

PI and collaborators have implemented the recurrent patterns program in detail [3] on a relatively simple system, 1-spatial dimension Kuramoto-Sivashinsky combustion equations that describe the flutter of a gas flame. For this specific problem many recurrent patterns have been determined numerically, and the periodic-orbit theory predictions tested for various values of flame front parameters. Kawahara and Kida [20] have demonstrated that the recurrent patterns can be determined in turbulent hydrodynamic flows by locating an important unstable spatio-temporally periodic solution in the three-dimensional plane Couette turbulence.

Both advances are a proof of principle, first steps in the direction of implementing recurrent patterns program. But there is a big conceptual gap to bridge between what has been achieved, and what needs to be done: Even the flame flutter has been probed only in its weakest turbulence regime, and it is an open question to what extent Hopf's vision remains viable as the system grows large and more turbulent.

7.1 Collaborative team

The program outlined above will require full time attention of one postdoctoral fellow and one or two Ph.D. students. The above Kuramoto-Sivashinsky and Navier-Stokes studies are proofs of principle, a first step in the direction of demonstrating the feasibility of the recurrent patterns program. Implementing the program in fluid dynamics requires computational power beyond PI's group's expertise. A project so ambitious will require that the PI puts together a strong collaborative team from natural sciences (physics, atmospheric sciences, oceanography) and engineering (aerospace, fluid dynamics) with complementary skills; experimental, numerical and theoretical.

The experience and skills of project team's **computational fluid dynamicists** are absolutely essential for any further progress. At Georgia Tech PI intends to ask F. Sotiropoulos, Assoc. Professor, School of Civil & Environmental Engineering and P.K. Yeung, Assoc. Professor, School of Aerospace Engineering, to join the project, and internationally PI is already a consultant on the current Kawahara-Kida proposal (Japan). **Experimental fluid dynamics** is pursued by a experimentalist-theorist team consisting of M. Schatz, Assoc. Professor, and R. Grigoriev, Assist. Professor, School of Physics, and another experimental biologist-computational fluid dynamicist team consisting of Professor Jeannette Yen, School of Biology, and F. Sotiropoulos.

Schatz group possesses a unique skill: by means of their multipoint thermal actuation technique, they are able to *design* spatio-temporal patterns, and thus create and test experimentally the patterns singled out by the theory. Thermally actuated control of thin films [25] pioneered here relies on modern electronics being much faster than fluid dynamics. The Schatz group, with expertise in real-time fluid control, can both observe [9] and simulate small fluid systems in real time. Were the theoretical models of comparable sophistication, this strategy could be reversed, and the real-time simulations could be used to *control* spatio-temporal patterns.

J. Yen's team has the ability to resolve experimentally 3-d whirls and whorls, and needs to quantify the dynamics underlying interaction of coherent structures in turbulent flows in real time; the team will address these challenges relying on experiments such as the real-time 3D optical measurements of wakes in fluids, and fluid dynamic simulations of such wakes [26].

7.2 Research plan

To the extent that such plans are ever realistic, here are the proposed research mileposts:

Year 1:

(a) **Variational searches for recurrent patterns:** Devise and implement an inversion of the loop linearized stability restricted to a finite inertial manifold subspace (regardless of the dimension of the discretized PDE truncation).

(b) **Kuramoto-Sivashinsky:** Test the variational code on 1-d Kuramoto-Sivashinsky flow for system sizes larger than those attained so far. The antisymmetric subspace (8) is unphysical, so explore the dynamics on the full space.

(c) **Symbolic dynamics:** Construct symbolic dynamics and approximate pruning rules for the above Kuramoto-Sivashinsky flow simulations, construct corresponding dynamical zeta functions, and compute expectation values of observables.

(d) **Plane Couette flow:** commence coding, testing the Navier-Stokes flow simulation code.

Year 2:

(a) **Kuramoto-Sivashinsky:** study equilibrium solutions for the 1-d system of infinite extent. Use this information to classify types of spatially periodic structures.

(b) **Systems of infinite spatial extent:** implement a variational search method for recurrent patterns, compute a set of shortest such patterns in 1-d Kuramoto-Sivashinsky system.

(c) **Plane Couette flow:** reproduce the Kawahara-Kida unstable solution for plane Couette turbulence, but now also compute its linearized stability.

Year 3:

(a) **Plane Couette flow:** Search for a hierarchy of longer periodic orbits, and their linearized stability, eigenspectra and and eigenvalues. Compute the frictional drag for a range of Re.

7.3 Intellectual merit

We already possess unique tools which we hope will lead to deterministic predictive capabilities for turbulence more quantitatively precise and generally applicable than many of the approaches currently pursued in fluid dynamics and engineering literature. The periodic orbit theory provides the machinery that converts the intuitive recurrent patterns picture into a precise calculational scheme.

The point is, although the initial numerical computation required can be intensely time, memory and CPU consuming, once an approximate finite alphabet of patterns and a vocabulary of realizable short words (patterns explored sequentially by the dynamics) has been extracted - and this is done offline - the result is an alphabet of order of 100-1000 of patterns. The dynamics in terms of these patterns might be fast enough that it can be implemented in real time - a prerequisite for applying this theory to engineering problems, such as the turbulent drag control and reduction.

The impact of a major advance here is by no means restricted to hydrodynamical turbulence. The key concepts should be applicable to many systems extended in space, and a successful theory of spatially extended systems would have broad impact, from problems involving motions of fluids to subatomic phenomena to assemblies of neurons. In a parallel (many degrees-of-freedom) but distinct (Hamiltonian, hence no low-dimensional attracting sets) PI group's research effort, NSF pending proposal #0355349, the semi-classical quantization of classically turbulent field theories, classical recurrent pattern solutions will be needed to implement any Gutzwiller-type trace formula semi-classical approximation to the quantum theory.

This proposal will address the grand challenge of nonlinear science: Explore experimentally and describe theoretically the dynamics of high-dimensional nonlinear systems. Furthermore, this proposal will foster applications of the methods thus developed to problems in engineering.

In E. Hopf's own words: *The ultimate goal, however, must be a rational theory of statistical hydrodynamics where [$\cdots$] properties of turbulent flow can be mathematically deduced from the fundamental equations of hydromechanics.*

7.4 Broader Impact

A modern education in the tools and methods of nonlinear science requires training that bridges traditional discipline boundaries. Students will acquire both the mathematical tools and develop physical intuition needed to tackle complex nonlinear problems arising in many different scientific fields. The Georgia Tech *Center for Nonlinear Science* [24] environment will complement the research component with a broad range of activities: interdepartmental research seminars, student-run seminars, an active visiting scientist program, and close interactions with Georgia Tech groups working on related problems [27, 28, 29], such as pattern formation and control, high-dimensional dynamics, coherent structures in turbulent flows. Collaborative visits to project partners (C.P. Dettmann [30] - Bristol, G. Vattay - Budapest, and others) will provide additional training experience and opportunities, both domestic and abroad.

The outreach initiatives will include undergraduate research participation and an advanced nonlinear dynamics course. Currently under development by the `ChaosBook.org` cross-disciplinary team (particle physicists, mathematicians, nuclear physicists, condensed matter experimentalists, ...), this novel hyper-linked web-based advanced graduate course [8] is already reaching students across the globe.

8 Prior, current and pending support

PI's nonlinear dynamics contributions include the Feigenbaum-Cvitanović universal equation for period doubling [31], the theory of cycle expansions [8], and applications of the theory to systems that exhibit classical and quantum chaos. PI is currently Glen Robinson Chair in Nonlinear Sciences and director of the newly created Georgia Tech *Center for Nonlinear Science* (CNS).

Prior to moving to US, Cvitanović founded and directed in the period 1993-1998 *Center for Chaos and Turbulence Studies* (CATS) at the Niels Bohr Institute, Copenhagen, a cross-disciplinary effort which became one of Europe's leading centers for nonlinear science, housing and in part funding approximately 15 faculty, 8 post-docs, 45 graduate students, 15 long term visitors, 40 short term visitors, and 5 workshops/conferences in any given year. As a recent arrival to US, he has no NSF support. However, in the period 1997-2000, prior to moving to Georgia Tech, P. Cvitanović led the initiative to create a Center for Complex Systems at the Northwestern University, and was the original PI on the IGERT #9987577: *Complex Systems in Science and Engineering program*, awarded to Northwestern for the period 2000-2004.

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