Kuramoto-Sivashinsky turbulence: a fishing expedition

Matt Marshall, *et al.*

May 6, 2007

Field trip by the Georgia Tech PHYS 4267/7224 'Introduction to nonlinear dynamics and chaos' class, spring semester 2007

1 Fluttering flame front

Turbulent flow drives a fluid through a repertoire of unstable patterns. As we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution, for a finite time the system follows approximately a pattern belonging to a finite alphabet of admissible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns, just as chaotic dynamics with a low dimensional attractor can be thought of as a succession of nearly periodic (but unstable) motions.

Here we apply this vision to the flame flutter of gas burning on your kitchen stove. We are happy if in a few days of experimentation we succeed in simulating the system numerically, and develop some intuition about turbulence.

2 Problem formulation

Please read the chapter "Turbulence?" [1]

A flame front is described by the Kuramoto-Sivashinsky [KS] equation

$$u_t = -\frac{1}{2}(u^2)_x - u_{xx} - u_{xxxx} \,. \qquad (1)$$

Here $t \geq 0$ is the time and $x \in [0, L]$ is the periodic space coordinate. In what follows we use interchangeably the "dimensionless system size" $\tilde{L}$, or the periodic domain size $L = 2\pi\tilde{L}$, as the system parameter. The subscripts x and t denote the partial derivatives with respect to x and t; $u_t = du/dt$, u_{xxxx} stands for 4th spatial derivative of the "velocity of the flame front" $u = u(x,t)$ at position x and time t.

As the "flame front velocity" $u(x,t) = u(x+2\pi,t)$ is periodic on the $x \in [0,2\pi]$ interval, expand it in a spatial Fourier basis:

$$u(x,t) = \sum_{k=-\infty}^{+\infty} a_k(t) e^{ikx/\tilde{L}} . \tag{2}$$

Since $u(x,t)$ is real,

$$a_k = a_{-k}^* . \tag{3}$$

Substituting (2) into (1) yields the infinite ladder of evolution equations for the complex Fourier coefficients $a_k(t)$:

$$\dot{a}_k = v_k(a) = ((k/\tilde{L})^2 - (k/\tilde{L})^4)\, a_k - i\frac{k}{2\tilde{L}} \sum_{m=-\infty}^{+\infty} a_m a_{k-m} . \tag{4}$$

As $\dot{a}_0 = 0$, the solution integrated over space is constant in time. We set this average velocity to zero, $a_0 = \int dx\, u(x,t) = 0$. The coefficients a_k are in general complex functions of time. The constant solution $u(x,t) = 0$ is an equilibrium point of (1). For this "laminar" equilibrium the stability matrix is diagonal,

$$A_{kj}(a) = \left(k^2/\tilde{L}^2 - k^4/\tilde{L}^4\right) \delta_{kj} , \tag{5}$$

and so is the Jacobian matrix $\mathbf{J}^t_{kj} = \delta_{kj} e^{(k/\tilde{L})^2(1-(k/\tilde{L})^2)t}$. From (5) it follows that the $|k| < \tilde{L}$ long wavelength modes of this equilibrium are linearly unstable, and the $|k| > \tilde{L}$ short wavelength modes are stable. For $\tilde{L} < 1$, $u(x,t) = 0$ is the globally attractive stable equilibrium, *i.e.*, the dissipation is so strong that any flame front burns out.

2.1 Energy budget

A theory of turbulence should predict measurable properties of turbulent flows, such as their mean energies and their energy dissipation rates.

The time-dependent average velocity-squared

$$E = \frac{1}{L} \int_0^L dx\, \frac{u^2}{2} \tag{6}$$

has a physical interpretation as the average "kinetic energy" density of the flame front. Its time evolution is given by the power/dissipation energy rate equation

$$\dot{E} = P - D , \qquad P = \left\langle (u_x)^2 \right\rangle , \quad D = \left\langle (u_{xx})^2 \right\rangle . \tag{7}$$

KS is a far-from equilibrium system: the power P pumped in by the anti-diffusion u_{xx} is balanced by the hypervicosity u_{xxxx} dissipation rate D. In principle, these are experimentally observable quantities, used in what follows as flow diagnostics.

2.2 Computation

We used R. L. Davidchack's implementation of Kassam and Trefethen code[2]:

`ChaosBook.org/extras/#PDEs` .

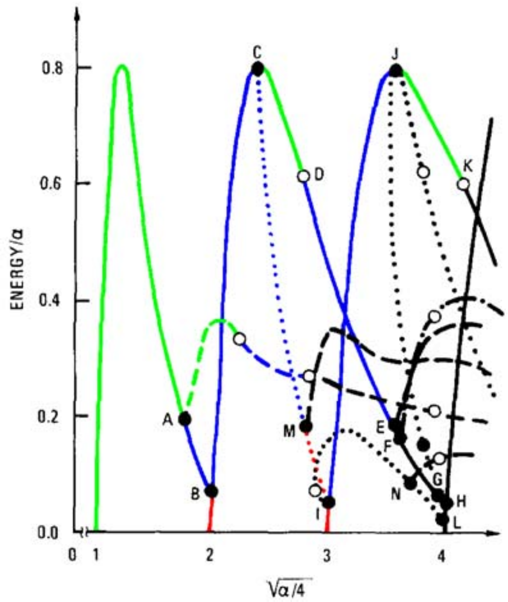

Figure 1: The energy (6) of all equilibria that exist up to $\tilde{L} = 4.5$ plotted as a function of the system size $\tilde{L}$ (from ref. [3]). Solid curves denote n-cell solutions, dotted curves GLMRT, the dash-dotted curve the 'giant states', and dashed curves the relative equilibria. Open circles indicate Hopf bifurcations. The color of a branch indicates the number of unstable eigenvalues: (red) 2 unstable eigenvalues, (blue) 1 unstable eigenvalue, (green) stable.

3 Fishing

From here on we turn to numerical experimentation. Take L sufficiently large so that the dynamics can be spatiotemporally chaotic, but not so large that we would be overwhelmed by many short wavelength modes needed in order to accurately represent the dynamics.

A bit of advice: start on *terra firma*, small system size $\tilde{L} = 1$, and increase $\tilde{L}$ a little bit, integrate until the trajectory has settled down; then increase $\tilde{L}$ a little bit again, restart from the trajectory just computed, integrate until has settled down. Repeat. Sometimes stop incrementing the trajectory, increment N instead and check how sensitive is your attractor to truncation number N. This "adiabatic" approach has advantage of (almost) always starting you close to the attractor, thus avoiding long transients typical of random starting conditions.

The problem with high-dimensional truncations of (4) is that the dynamics is difficult to visualize. We visualize a trajectory by its projections onto any three 2 or 3 basis vectors (states of the system): see Davidchack code for examples.

3.1 Bifurcations with the increase of system size L

Explore dynamics for different system sizes L by plotting long trajectories both in the state space and in color-coded space-time plots of $u(x, t)$. If the long time trajectory is settling on an attractive equilibrium or periodic orbit, eliminate the initial transient by starting the plots only after the transients have died out.

Classical papers on the equilibria of Kuramoto-Sivashinsky equation, bifurcations and symmetry analysis are Kevrekidis, Nicolaenko and Scovel [4], and Greene and Kim [3]. You might be able to identify your states in the fig. 3 bifurcation diagram. To get the right E, multiply by 4; the horizontal axis is $\tilde{L} = \sqrt{\alpha/4}$.

Here is what the class fished out, listed as series of numerical experiments ordered by $\tilde{L}$ parameter. Please send you .png files to Matt Marshall, `raequin[snail]gmail.com`

1.5915494: $L = 10$ (Holt)
stable E_1

1.7603981: $\tilde{L} = 1 + \pi/4 - 0.025$ (Marshall)
What's this bifurcation? Narrow the gap ± 0.025 to see whether it really happens at $\tilde{L} = 1 + \pi/4 \pm \epsilon$. Why would that happen?

1.8103981: $\tilde{L} = 1 + \pi/4 + 0.025$ (Marshall)

1.9098593: $L = 12$ (Holt, Hermand)
stable periodic orbit

2: $\tilde{L} = 2$ (Marshall)

2.0690142: $L = 13$ (Iacobucci)
relative periodic?

2.078125: $\tilde{L} = 2 + 5/64$ (Marshall)
slowly drifting relative periodic?

2.09375: $\tilde{L} = 2 + 3/32$ (Marshall)
flipping between E_1 and 1/2 shifted E_1?

2.1591549: $\tilde{L} = 2 + 1/2\pi$ (Marshall)
flipping between E_1 and 1/2 shifted E_1?

2.2281692: $L = 14$ (Hermand)
stable E_2
item[2.5:] $\tilde{L} = 2 + 1/2$ (Marshall)
why does this trajectory project to $(0, 0, 0)$ in Ruslan's (v_1, v_2, v_3) basis?

2.8647882: $L = 18$ (Marshall, Hermand)
attractive cycle?

3: $L = 3$ (Sonenblum)
flipping between E_1 and 1/2 shifted E_1?

3.0239439: $L = 19$ (Holt)
stable E_2

3.22: $\tilde{L} = 2.5 + 3/4 - 0.03$ (Marshall)
attractive cycle

3.28: $\tilde{L} = 2.5 + 3/4 + 0.03$ (Marshall)
flipping between E_2 and 1/2 shifted E_2, via the above unstable cycle neighborhood? What's this bifurcation? Narrow the gap ± 0.03 to see whether it really happens at $\tilde{L} = 2.5 + 3/4 \pm \epsilon$

3.3422538: $L = 21$ (Li)
relative periodic attractor? unclear

3.6605636: $L = 23$ (Davydychev, Holt)
strange - looks like cycle but keeps flipping

3.8197186: $L = 24$ (Hermand, Silverman)
spiral into E_3 attractor

3.8408862: $L = 24.133$ (Marshall)
spiral into E_3 attractor

3.978873: $L = 25$ (Li, Hermand)
spiral into E_3 attractor

4.138028: $L = 26$ (Anzalone, Iacobucci, Holt)
unclear. Iacobucci drifting 3-cycle, Holt 3-cycle

4.4563384: $L = 28$ (Davydychev)
equilibrium

4.6154933: $L = 29$ (Hermand)
settles into a wierd state, asymptotic only after $t > 250$

5: $L = 5$ (Sonenblum)
stable E_4

5.411268: $L = 34$ (Freeman)
co-existing attractors?

6.3661977: $L = 40$ (Iacobucci)
interesting transient

7: $L = 7$ (Sonenblum)
turbulent

9: $L = 9$ (Sonenblum)
turbulent

3.2 Energy dependence on L

Iacobucco and Freeman have interesting ways of scanning energy as function of L. With this they have explored bifurcation sequences by plotting (*inter alia*) the time-averaged values $(\overline{E}, \overline{P}, \overline{D})$ of (E, P, D) defined in (7), for small increments of $1 < L < 25$.

3.3 Turbulence at $\tilde{L} = 22$?

1. **Plot** several long trajectories for $L = 22$, different initial conditions, using the same vector basis as Davidchack. Is your dynamics qualitatively the same as in his plots?

   [Holt has nice long time runs]

2. Do you get any stable periodic orbits for $\tilde{L} = 22$? If you do, we would love to see them - have not found any.

   [Nobody saw anything different, and no stable orbits were observed. Only Sharon tried a set of different initial conditions, but al close-by]

4 Turbulence goes Hollywood

Kirill Davydychev: I left my laptop overnight to compute this flame front movie:

FlameFront [avi format].

From it one can guess the stable/unstable cycles, the period for them seems to be around $7L$, with $3.5\,L$ separation.

you mean period in time, or that by increasing $\tilde{L}$ by 1 you see another bifurcation?

Also, here is a geeky approach to the onset of chaos: consider the file sizes of each frame. In a lossless format such as .png, the compression ratio is an indicator of the complexity of the structure, see graph of L vs. the file size. It correlates with the chaotic behavior of this system (not to mention that it's also completely useless).

Now, have a Carlsberg, perhaps the best beer in some parts of Copenhagen, and a good summer.

References

[1] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay. *Chaos: Classical and Quantum.* Niels Bohr Institute, Copenhagen, 2005. `ChaosBook.org`.

[2] A. K. Kassam and L. N. Trefethen. Fourth-order time stepping for stiff PDEs. *SIAM J. Sci. Comp.*, 2004.

[3] J. M. Greene and J. S. Kim. The steady states of the Kuramoto-Sivashinsky equation. *Physica D*, 33:99–120, 1988.

[4] I. G. Kevrekidis, B. Nicolaenko, and J. C. Scovel. Back in the saddle again: a computer assisted study of the Kuramoto-Sivashinsky equation. *SIAM J. Appl. Math.*, 50(3):760–790, 1990.