Exercise 8.3 A limit cycle with analytic stability exponent. *There are only two examples of nonlinear flows for which the stability eigenvalues can be evaluated analytically. Both are cheats. One example is the 2-d flow*

$$\dot{q} = \ p + q(1 - q^2 - p^2) \, , \qquad \dot{p} = -q + p(1 - q^2 - p^2) \, . \tag{8.24}$$

Determine all periodic solutions of this flow, and determine analytically their stability exponents. Hint: go to polar coordinates $(q, p) = (r \cos \theta, r \sin \theta)$.

G. Bard Ermentrout

Exercise 8.4 The other example of a limit cycle with analytic stability exponent. *What is the other example of a nonlinear flow for which the stability eigenvalues can be evaluated analytically? Hint: email G.B. Ermentrout.*

Exercise 8.5 Yet another example of a limit cycle with analytic stability exponent. *Prove G.B. Ermentrout wrong by solving a third example (or more) of a nonlinear flow for which the stability eigenvalues can be evaluated analytically.*

Chapter 8

Solution 8.3: A limit cycle with analytic stability exponent. *The 2-d flow (8.24) is cooked up so that $x(t) = (q(t), p(t))$ is separable (check!) in polar coordinates $q = r\cos\phi\,,\quad p = r\sin\phi$:*

$$\dot{r} = r(1-r^2)\,, \qquad \dot{\phi} = 1\,. \tag{N.9}$$

In the (r,ϕ) coordinates the flow starting at any $r > 0$ is attracted to the $r = 1$ limit cycle, with the angular coordinate ϕ wraping around with a constant angular velocity $\Omega = 1$. The non–wandering set of this flow consists of the $r = 0$ equilibrium and the $r = 1$ limit cycle.

equilibrium stability: *As the change of coordinates is defined everywhere except at the the equilibrium point ($r = 0$, any ϕ), the equilibrium stability matrix (4.26) has to be computed in the original (q,p) coordinates,*

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{N.10}$$

The eigenvalues are $\lambda = \mu \pm i\,\nu = 1 \pm i$, indicating that the origin is linearly unstable, with nearby trajectories spiralling out with the constant angular velocity $\Omega = 1$. The Poincaré section ($p = 0$, for example) return map is in this case also a stroboscopic map, strobed at the period (Poincaré section return time) $T = 2\pi/\Omega = 2\pi$. The radial stability multiplier per one Poincaré return is $|\Lambda| = e^{\mu T} = e^{2\pi}$.

Limit cycle stability: *From (N.9) the stability matrix is diagonal in the (r,ϕ) coordinates,*

$$A = \begin{bmatrix} 1-3r^2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{N.11}$$

The vanishing of the angular $\lambda_\theta = 0$ eigenvalue is due to the rotational invariance of the equations of motion along ϕ direction. The expanding $\lambda_r = 1$ radial eigenvalue of the equilibrium $r = 0$ confirms the above equilibrium stability calculation. The contracting $\lambda_r = -2$ eigenvalue at $r = 1$ decreases the radial deviations from $r = 1$ with the radial stability multiplier $\Lambda_r = e^{\mu T} = e^{-4\pi}$ per one Poincaré return. This limit cycle is very *attracting.*

Stability of a trajectory segment: *Multiply (N.9) by r to obtain $\frac{1}{2}\dot{r^2} = r^2 - r^4$, set $r^2 = 1/u$, separate variables $du/(1-u) = 2\,dt$, and integrate: $\ln(1-u) - \ln(1-u_0) = -2t$. Hence the $r(r_0,t)$ trajectory is*

$$r(t)^{-2} = 1 + (r_0^{-2} - 1)e^{-2t}\,. \tag{N.12}$$

The $[1\times 1]$ *fundamental matrix*

$$J(r_0,t) = \left.\frac{\partial r(t)}{\partial r_0}\right|_{r_0=r(0)} . \tag{N.13}$$

satisfies (4.33)

$$\frac{d}{dt}J(r,t) = A(r)\,J(r,t) = (1-3r(t)^2)\,J(r,t)\,, \qquad J(r_0,0) = 1\,.$$

This too can be solved by separating variables $d(\ln J(r,t)) = dt - 3r(t)^2 dt$ *, substituting (N.12) and integrating. The stability of any finite trajectory segment is:*

$$J(r_0,t) = (r_0^2 + (1-r_0^2)e^{-2t})^{-3/2}e^{-2t}\,. \tag{N.14}$$

On the $r=1$ *limit cycle this agrees with the limit cycle multiplier* $\Lambda_r(1,t) = e^{-2t}$*, and with the radial part of the equilibrium instability* $\Lambda_r(r_0,t) = e^t$ *for* $r_0 \ll 1$*.*

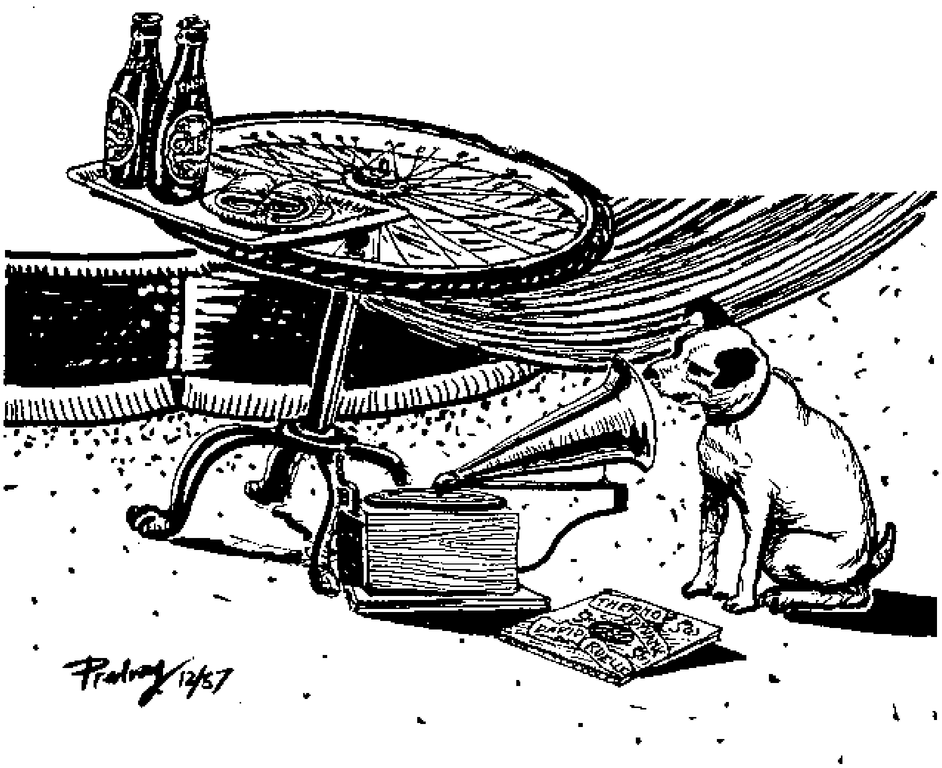
P. Cvitanović

Solution 8.4: The other example of a limit cycle with analytic stability exponent. *Email your solution to ChaosBook.org and G.B. Ermentrout.*

Solution 8.5: Yet another example of a limit cycle with analytic stability exponent. *Email your solution to ChaosBook.org and G.B. Ermentrout.*

Chaos: Classical and Quantum

Part I: Deterministic Chaos



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ChaosBook.org/version11.9, Dec 4 2006 printed February 2, 2007

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