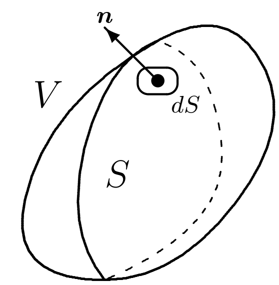
6

Planets and stars

Planets and stars are objects held together by their own gravity, but prevented from collapsing by internal pressure, originating from repulsive atomic or nuclear forces. The more massive a body is, the higher is the pressure necessary to prevent collapse. For sufficiently massive bodies the ultimate gravitational collapse cannot be prevented by any known forces, and will eventually occur, and a black hole is born.

So far we have only been able to scratch the surface of our own planet Earth. A little has also been done on the Moon and soon we shall know more about the surface of Mars. Seismic waves created by controlled explosions do allow us to peer deeper into the planet, but mostly we are left with the "experiments" carried out by nature without any regard to us. Earthquakes generate strong seismic waves, revealing the inner structure of the planet. Continental drift informs us about the mixture of heat and gravity deep inside. Electromagnetic radiation from the surface of a star is almost the only source of information about what goes on below, although neutrino observations have begun to provide a direct window into the deepest core of our Sun, and into the supernovas that explode in our cosmic neighborhood.

Most of our understanding of the interiors of planets and stars comes from using the laws of physics determined on Earth as an "analytic drill" allowing us to get insight into processes which cannot be directly observed from the outside. In this chapter, the first turns of this drill consist in applying the equations of hydrostatic equilibrium to these massive self-gravitating bodies. The strongest simplifying assumption we can make about planets and stars, is that they are spherically symmetric, but before we specialize to that case we need to derive a fundamental differential equation connecting a mass distribution and its gravitational field. At the end of the chapter, we apply the formalism to an isentropic star without internal energy production.



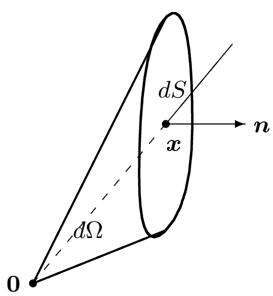
A surface S surrounding a volume V. The direction $\boldsymbol{n}$ of a surface element dS is always oriented outwards from the volume.

6.1 Gravitational flux

Let S be a closed surface surrounding a volume V. We shall as before use the convention that the normal $\boldsymbol{n}$ in a point $\boldsymbol{x}$ of the surface is always a unit vector oriented *outwards* from the surface, and a small surface element of magnitude ds is represented by the vector $d\boldsymbol{S} = \boldsymbol{n}dS$.

Seen from the origin of the coordinate system, the solid angle subtended by this surface element is

$$d\Omega = \frac{\boldsymbol{x} \cdot d\boldsymbol{S}}{|\boldsymbol{x}|^3} \ . \tag{6-1}$$



The solid angle subtended by a surface element

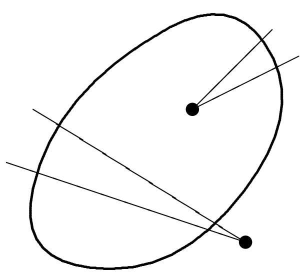
Projecting the gravitational field (3-12) from a point mass on the surface element, one obtains

$$\boldsymbol{g} \cdot d\boldsymbol{S} = -GMd\Omega \ .$$

This quantity is called the *flux of gravity* through the surface element.

Consider now the total flux through a closed convex surface containing the point mass at the origin. In that case, all the little solid angles add up to 4π because the line-of-sight from the particle in any direction crosses the convex surface exactly once. If on the other hand the surface does not contain the point mass, the line of sight from the particle will always cross the surface twice, and the two contributions to the solid angle will have the same magnitude but opposite sign and thus cancel. In other words,

$$\oint_S \boldsymbol{g} \cdot d\boldsymbol{S} = \begin{cases} -4\pi GM & \text{for } \boldsymbol{0} \in V \\ 0 & \text{otherwise} \end{cases} \ .$$



The lines of sight from a point inside a convex volume crosses the surface once, whereas they cross twice if the mass is outside.

This result is in fact valid for all surfaces, convex or not. For a convoluted surface, the line-of-sight from the inside will instead cross the surface an odd number of times, and since the solid angles are evaluated with sign, all the contributions along the line-of-sight cancel each other except for one. If the particle is outside the volume the line-of-sight will cross an even number of times and all contributions cancel. The conclusion is that the above equation holds in full generality.

Furthermore, this result cannot depend on the particle being at the origin, but must be generally valid for any point particle inside or outside the volume. Adding together the contributions from all the material particles in the volume V, we finally get

$$\boxed{\oint_S \boldsymbol{g} \cdot d\boldsymbol{S} = -4\pi G \int_V \rho \, dV} \ . \tag{6-2}$$

The integral at the right is simply the total mass (3-2) in the volume, so we may conclude that the gravitational flux through any closed surface is proportional to the total mass contained within the surface, whereas the mass outside the surface does not contribute to the flux.

Gauss' theorem and divergence

We have previously derived a vector relation (4-20) between a surface integral if a scalar field and a volume integral over its gradient. Applying it componentwise to the left hand side of (6-2) we obtain

$$\oint_S \boldsymbol{g} \cdot d\boldsymbol{S} = \oint_S (g_x dS_x + g_y dS_y + g_z dS_z) = \int_V (\nabla_x g_x + \nabla_y g_y + \nabla_z g_z)\, dV \ .$$

This is the usual form of Gauss' theorem

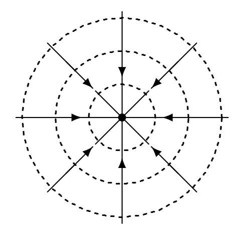
$$\boxed{\oint_S \boldsymbol{g} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{g} \ dV} \ , \qquad (6\text{-}3)$$

where the field on the right hand side

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = \nabla_x g_x + \nabla_y g_y + \nabla_z g_z = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} \ , \qquad (6\text{-}4)$$

is called the *divergence* of the gravitational field[1]. Its value in a point is a measure of how much field lines diverge away from each other, or converge if it is negative.

This form of Gauss' theorem is also a relation between any vector field $\boldsymbol{g}$, not necessarily the gravitation field, and its divergence $\boldsymbol{\nabla} \cdot \boldsymbol{g}$. The two forms, (4-20) and (6-3), are equivalent (see problem 6.1).



The minus sign in the source equation (6-5) expresses that field lines always converge upon masses.

Poisson's equation

The global equation (6-2) relating the gravitational field to its sources may now, like the global hydrostatic equation (4-14), be converted to a local differential equation. Using Gauss' theorem (6-3) we find from (6-2)

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{g}\, dV = -4\pi G \int_V \rho\, dV \ ,$$

which must be valid for all volumes V. That is, however, only possible, if integrands are equal, or

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G \rho} \ . \qquad (6\text{-}5)$$

This is one of the fundamental field equations of gravity, expressing that the mass density is the local *source* of the gravitational field.

It is convenient to define the *Laplace operator*

$$\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \nabla_x^2 + \nabla_y^2 + \nabla_z^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \qquad (6\text{-}6)$$

Pierre Simon marquis de Laplace (1749–1827). *French mathematician, astronomer, and physicist. Developed gravitational theory and applied it to perturbations in the planetary orbits and the conditions for stability of the solar system.*

[1] In the older literature the divergence of a vector field was denoted $\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v}$. In modern times this notation has been abandoned, and so shall we.

This operator plays a major role in all field theories.

Using that $\boldsymbol{g} = -\nabla\Phi$ (see section 3.4), the source equation (6-5) may be rewritten in terms of the gravitational potential, and we obtain *Poisson's equation*,

$$\boxed{\nabla^2\Phi = 4\pi G\rho} \ . \tag{6-7}$$

Simeon Denis Poisson (1781–1840). *French mathematician. Contributed to electromagnetism, celestial mechanics, and probability theory.*

The linearity of this equation guarantees that if Φ_1 is a particular solution then the most general solution is of the form $\Phi = \Phi_0 + \Phi_1$ where Φ_0 is an arbitrary solution to

$$\nabla^2\Phi_0 = 0 \ , \tag{6-8}$$

also called *Laplace's equation*. The actual solution selected in a particular problem depends on the boundary conditions.

Example 6.1.1: If the universe were uniformly filled with matter at constant density, $\rho(\boldsymbol{x}) = \rho_0$, we would have

$$\nabla^2\Phi = 4\pi G\rho_0 \ . \tag{6-9}$$

It is easy to verify explicitly that a particular solution to this equation is

$$\Phi = \frac{2}{3}\pi G\rho_0 \left(x^2 + y^2 + z^2\right) \ , \tag{6-10}$$

corresponding to a gravitational acceleration

$$\boldsymbol{g} = -\frac{4}{3}\pi G\rho_0 \boldsymbol{x} \ . \tag{6-11}$$

The gravitational field points everywhere towards the origin of the coordinate system which is thus imbued with an apparently unphysical preferred status. In section 14.6 we shall nevertheless see that this field comes in handy in a Newtonian model for cosmology.

Hydrostatic equilibrium

One may rightly ask why we need Poisson's equation when the complete connection between a mass distribution and its gravitational potential is already given by the integral (3-24). For compressible matter, however, the mass density depends on the pressure, which in turn depends on gravity through the equation of hydrostatic balance (4-18), and gravity depends in its turn on the mass density. Such physical circularity is best handled by means of differential equations.

To see how this works out, we use (4-18) and (6-5) to calculate the divergence of $\boldsymbol{g} = \nabla p/\rho$, and obtain

$$\boxed{\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = -4\pi G\rho} \ . \tag{6-12}$$

Together with an equation of state of the form $p = p(\rho)$, this becomes a nonlinear, second order partial differential equation for the density field (or the pressure).

6.2 Spherical bodies

The mass distribution $\rho(r)$ for spherically symmetric body such as a planet or a star is, as discussed in section 3.3, only a function of the distance $r = |\boldsymbol{x}|$ from its center, which is taken to be at the origin of the coordinate system. The field of gravity must correspondingly be radial, $\boldsymbol{g}(\boldsymbol{x}) = g(r)\,\boldsymbol{e}_r$, with $\boldsymbol{e}_r = \boldsymbol{x}/r$. Applying the global source equation (6-2) to a spherical surface $S(r)$ of radius r, the surface integral on the left hand side becomes

$$\oint_{S(r)} \boldsymbol{g} \cdot d\boldsymbol{S} = 4\pi r^2 g(r) \ , \tag{6-13}$$

because the surface area of the sphere is $4\pi r^2$. The volume integral on the right hand side of (6-2) is simply the integrated mass $M(r)$ given in (3-18), so that we obtain

$$g(r) = -\frac{GM(r)}{r^2} \ . \tag{6-14}$$

Finally, we have fulfilled the promise of deriving eq. (3-17).

The general equation of hydrostatic equilibrium (6-12) simplifies considerably for a spherical system, and becomes an ordinary second order differential equation for the pressure $p(r)$ or the density $\rho(r)$. Instead of trying to derive this differential equation from the general one, it is easier to go back to the original equation of local hydrostatic equilibrium (4-18). Since

$$\boldsymbol{\nabla} p(r) = \frac{dp(r)}{dr}\boldsymbol{\nabla} r = \frac{dp(r)}{dr}\boldsymbol{e}_r$$

we get

$$\boxed{\frac{dp(r)}{dr} = g(r)\rho(r) = -G\frac{M(r)}{r^2}\rho(r)} \ . \tag{6-15}$$

Multiplying with r^2/ρ and differentiating after r, we find

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -G\frac{dM(r)}{dr} = -G4\pi r^2\rho(r) \ ,$$

and rearranging, this becomes

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho} \ . \tag{6-16}$$

Combined with an equation of state of the form $p = p(\rho)$, this is an ordinary second order differential equation for the density or pressure. In fig. 6.1 the Earth's pressure distribution is plotted and compared with a simple model.

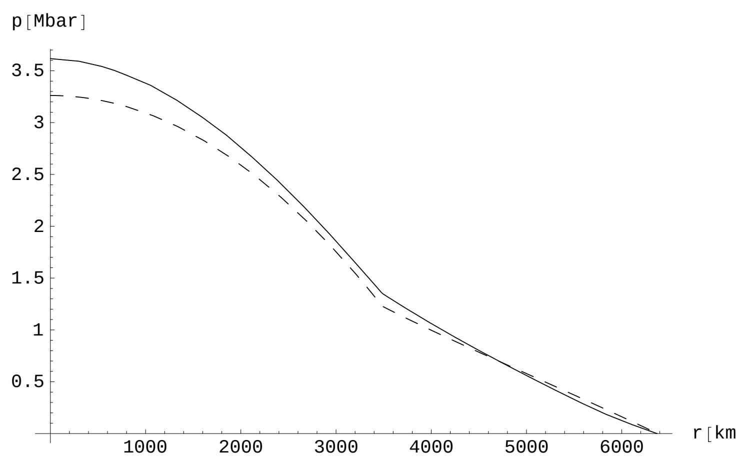


Figure 6.1: *Pressure distribution in the Earth. Fully drawn: data from [3] and dashed: the two-layer model (problem 6.7). The agreement between the model and data is impressive in view of the coarseness of the model.*

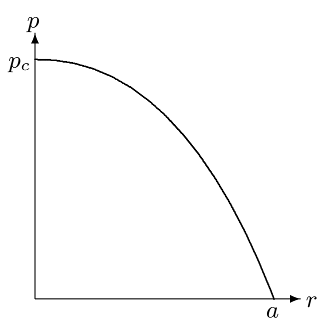
Boundary conditions

In principle, a second order differential equation requires two boundary values (or integration constants), for example the central pressure $p_c = p(0)$ and its first derivative dp/dr for $r = 0$. We shall make the reasonable assumption that the density ρ_c at the center of the body is finite. Then for 'small' r we have $M(r) \approx \frac{4}{3}\pi r^3 \rho_c$ and eq. (6-15) becomes for $r \to 0$,

$$\frac{dp}{dr} \approx -\frac{4}{3}\pi G \rho_c^2 r \ ,$$

which integrates to

$$p(r) \approx p_c - \frac{2}{3}\pi G \rho_c^2 r^2 \ . \qquad (6\text{-}17)$$



The pressure varies as a parabola in the central region of a spherically invariant body with a finite central density.

Thus, under the assumption of finite central density, the pressure is parabolic near the center with $dp/dr = 0$ for $r = 0$. This shows that under reasonable physical assumptions the hydrostatic equation (6-16) requires in fact only one boundary condition, for example the central pressure. Knowing p_c together with the equation of state (which also determines ρ_c), the pressure may be calculated throughout the body.

The central pressure and density are, of course, not known for planets and stars, objects that are only accessible from the outside. Most such bodies have a well-defined surface radius, $r = a$, at which the pressure vanishes. We shall arbitrarily call a body a planet, if the density i jumps abruptly to zero at the surface, and a star if the density vanishes along with the pressure at the surface. Such a convention makes the gaseous giant planets, Jupiter and Saturn, count as stars even if they probably do not burn much hydrogen.

The requirement of zero pressure at $r = a$ will determine the central pressure. The solutions to the hydrostatic equation can be expressed entirely in terms of the radius of the body and the parameters in the equation of state. In particular the mass M_0 of the body is — as we shall see below — calculable in terms of a (and the state parameters). Conversely, if the mass and radius are known, one of the other unknown parameters may be determined.

Planet with constant density

For a planet with constant density, ρ_0, the finite central assumption (6-17) is exactly valid throughout the planet,

$$p = p_c - \frac{2}{3}\pi G \rho_0^2 r^2 \ . \tag{6-18}$$

At the surface of the planet where the pressure has to vanish this leads to

$$p_c = \frac{2}{3}\pi G \rho_0^2 a^2 \ . \tag{6-19}$$

If the mass and radius are known, the density is obtained from $M_0 = \frac{4}{3}\pi a^3 \rho_0$.

6.3 The homentropic star

Stars like the Sun are self-gravitating, gaseous, and almost perfectly spherical bodies, generating heat by thermonuclear processes in a fairly small region close to the center. The heat is transferred to the surface by radiation, conduction and convection and eventually released into space as radiation. Stars also have a fairly complex structure with several layers differing in chemical composition and other physical properties.

Example 6.3.1: Our Sun consists of a mixture of about 71% hydrogen, 27% helium, and 2% other elements. It has a central core of radius 150, 000 km, a radiative layer of thickness 350, 000 km, and a convection layer of thickness 200, 000 km. The "standard" values for the central parameters are [11] $T_c = 15.7 \times 10^6$ K, $\rho_c = 154$ g/cm^3, and $p_c = 2.34 \times 10^{11}$ bar.

The stellar temperature lapse rate

Here we shall completely ignore the layering, heat production and chemical composition, and concentrate solely on hydrostatic equilibrium in a homogeneous star. We shall assume that the whole star consists of an ideal gas with adiabatic index $\gamma = 5/3$, and molar mass $M_{\mathsf{mol}} = 0.5$ g/mol. This corresponds to fully ionized hydrogen, which consists af 50% hydrogen ions (protons) and 50% essentially massless electrons. Apart from a layer near the surface, the effective adiabatic index, defined by $1 - 1/\gamma_{\text{eff}} = d\log T/d\log p$ is in fact very close to this value throughout the Sun (see fig. 6.2 and problem 6.10).

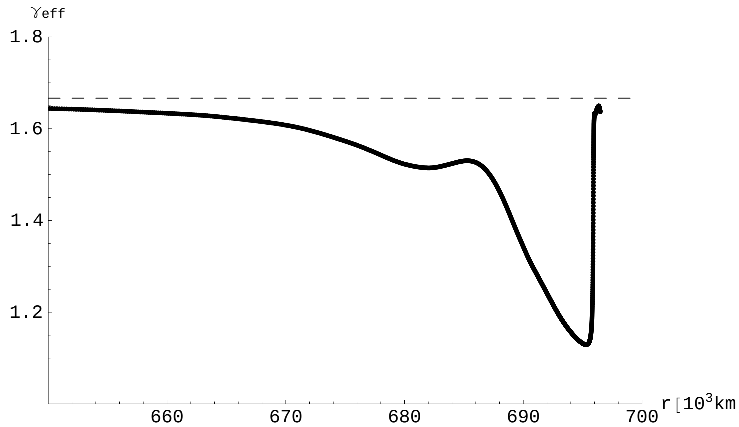


Figure 6.2: *The effective adiabatic index γ_{eff} near the surface of the Sun. The fully drawn curve is taken from the 'standard' Sun model [11] and the dashed line is the constant monatomic value $\gamma = 5/3$.*

In section 4.6 we argued (for the case of Earth's atmosphere) that provided the time scale for local mixing was fast compared to heat conduction, a *homentropic* dynamical "equilibrium" would be established, in which the specific entropy (4-54) took the same value everywhere. The same argument applies to the convective layers of a star, and we shall now assume that the whole star is homentropic with constant specific entropy everywhere in the star. From the constancy of $T^\gamma p^{1-\gamma}$ we obtain as in section 4.6,

$$\gamma \frac{1}{T}\frac{dT}{dr} + (1-\gamma)\frac{1}{p}\frac{dp}{dr} = 0 \ .$$

Using (6-15) we find the *stellar temperature lapse rate*,

$$\boxed{\frac{dT(r)}{dr} = \frac{g(r)}{c_p}} \ , \qquad (6\text{-}20)$$

where $g(r) = -GM(r)/r$ is the acceleration field (6-14), and $c_p = \gamma/(\gamma - 1)\ R/M_{\mathsf{mol}}$ is the specific heat (4-51) of the ideal gas at constant pressure. The only difference is that in the atmosphere the acceleration is constant, whereas in the star it depends on r.

The above equation may be converted to a second order differential equation,

$$\boxed{\frac{c_p}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -4\pi G\rho \ .} \qquad (6\text{-}21)$$

On the right hand side we must use the constancy of $T\rho^{1-\gamma}$ to eliminate the density and make it a differential equation for T only.

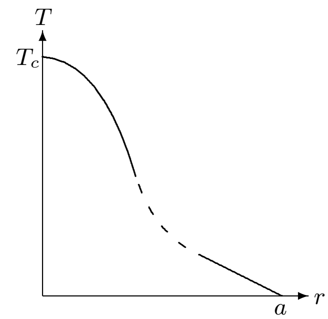
Approximative solutions near the center and the surface

There are many types of solutions to the stellar equations (6-20) or (6-21). Some have infinite central pressure, others have non-vanishing density all the way to infinity (see problem 6.13). We shall limit ourselves to solutions with finite central density and a well-defined radius where the density vanishes.

If the central density ρ_c is finite, the integrated mass becomes $M(r) \approx \frac{4}{3}\pi r^3 \rho_c$ near the center, and thus $g(r) \approx -\frac{4}{3}\pi G \rho_c r$. From (6-20) we then obtain,

$$T \approx T_c - \frac{2\pi}{3} \frac{G\rho_c}{c_p} r^2 \ , \tag{6-22}$$

where T_c is the central temperature. Evidently, the temperature drops as a parabola away from the center of the star, and in the leading approximation this is also true for the pressure and the density.



The temperature follows a parabola in the central region and approaches zero linearly near the surface. The dashed curve interpolates between these two extremes.

If the density vanishes at the surface, $r = a$, the temperature and pressure must also vanish. From (6-20) it follows that the temperature derivative is finite close to the surface at $r = a$, so that we may make a linear approximation

$$T(r) \approx T_0 \left(1 - \frac{r}{a}\right) \ , \tag{6-23}$$

near the surface. Inserting this into (6-20) and taking $r = a$ on the right hand side, we find

$$T_0 = \frac{g_0 \, a}{c_p} \ , \tag{6-24}$$

where $g_0 = GM_0/a^2$ is the magnitude of the star's surface gravity. Notice that this temperature which sets the scale of the temperature gradient at the surface is calculable in terms of the star's known parameters.

> **Example 6.3.2:** Putting in the Sun's parameters, $M_0 \approx 2\times10^{30}$ kg, $a \approx 7\times10^8$ m, and $c_p \approx 4.2 \times 10^4$ J/K/kg, we find $g_0 \approx 274$ m/s^2 and $T_0 \approx 4.6 \times 10^6$ K. Even if the surface approximation is not valid near the center, T_0 is nevertheless of the same magnitude as the Sun's central temperature.

Interpolation between center and surface

Having determined the behavior of the temperature near the center as well as near the surface, we need to interpolate between these regions. From the general discussion of boundary conditions in section 6.2, we expect that the stellar equation (6-21) will create a connection between the central temperature T_c and the calculable temperature parameter T_0.

Let us introduce the dimensionless variable $\xi = r/\lambda$, where λ is a suitable constant with the dimension of length, and the dimensionless temperature function

$$\theta(\xi) = \frac{T(r)}{T_c} \ . \tag{6-25}$$

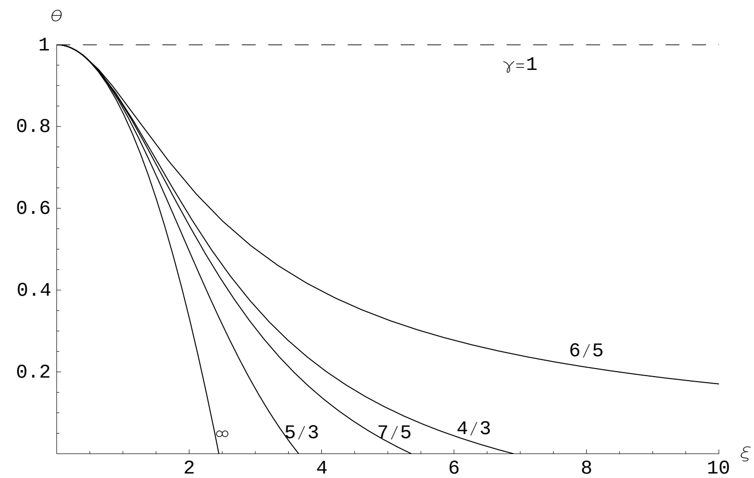


Figure 6.3: *The family of Lane-Emden functions for selected values of γ.*

The density is calculated from the homentropic condition $T\rho^{1-\gamma} = T_c\rho_c^{1-\gamma}$,

$$\rho = \rho_c\,\theta^{\frac{1}{\gamma-1}} \ , \tag{6-26}$$

Choosing the length parameter to be

$$\lambda = \sqrt{\frac{c_p T_c}{4\pi G\rho_c}} \ , \tag{6-27}$$

the stellar equation (6-21) becomes the *Lane-Emden equation*,

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^{\frac{1}{\gamma-1}} = 0 \ .} \tag{6-28}$$

γ	ξ_0	T_c/T_0
∞	*2.449*	*0.500*
5/3	*3.654*	*1.346*
7/5	*5.355*	*2.449*
4/3	*6.897*	*3.417*

Table of the crossing points ξ_0 and scaled central temperature T_c/T_0 for the Lane-Emden functions at selected values of γ.

From the solution near the center of the star we conclude that the boundary conditions for $\theta(\xi)$ are $\theta(0) = 1$ and $\theta'(0) = 0$. The solutions form a family of functions parameterized by the adiabatic index γ.

Apart from special cases (see below and problem 6.13) this differential equation cannot be solved analytically. In Fig. 6.3 the Lane-Emden functions have been evaluated numerically for a few relevant values of γ. For $\gamma > 6/5$ it may be shown that the solutions cross the ξ-axis. This means that θ vanishes at this point, which is identified with the boundary of the star and denoted $\xi_0 = \xi_0(\gamma)$. Its precise value may be calculated numerically for all $\gamma > 6/5$. A few relevant ones are given in the table in the margin.

The limiting cases of the Lane-Emden functions are easily determined analytically. For $\gamma \to 1$, corresponding to an isothermal star, the solution is $\theta(\xi) \to 1$ so that $T(r) = T_c$ for all r (with a jump at the surface, making the star into a planet, according to our definition). For $\gamma \to \infty$, we get from (6-18) and the ideal gas law, $\theta(\xi) \to 1 - \xi^2/6$, which also follows from (6-28). This curve crosses the axis at $\xi_0(\infty) = \sqrt{6}$.

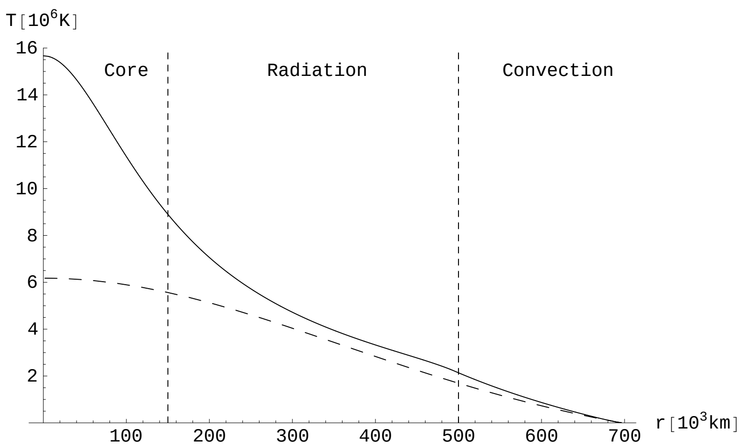


Figure 6.4: *The temperature distribution in the Sun as a function of the distance from the center. The fully drawn curve is from the "standard" Sun model [11] and the dashed curve is the Lane-Emden solution for $\gamma = 5/3$. The vertical lines are boundaries between various layers of the Sun.*

Central values

Knowing $\xi_0 = \xi_0(\gamma)$ the value of the scaling parameter $\lambda = a/\xi_0$ can be calculated from the known radius a of the star. Then from (6-20) at $r = a$ we get

$$\frac{T_c}{\lambda}\theta'(\xi_0) = -\frac{g_0}{c_p} ,$$

where $\theta'(\xi_0)$ is the slope of the solution at ξ_0. Introducing the temperature scale T_0 from (6-24) this becomes,

$$\frac{T_c}{T_0} = \frac{1}{(-\theta'(\xi_0))\xi_0} . \tag{6-29}$$

A few selected values are shown in the margin table.

Similarly from (6-27) we find the central density

$$\frac{\rho_c}{\rho_0} = \frac{\xi_0^2}{3}\frac{T_c}{T_0} , \tag{6-30}$$

where $\rho_0 = M_0/\frac{4}{3}\pi a^3$ is the average density of the star. Knowing both ρ_c and T_c allows us to determine the central pressure is found from the ideal gas law $p_c = \rho_c R T_c / M_{\mathsf{mol}}$.

Example 6.3.3: For $\gamma = 5/3$ we obtain $T_c/T_0 = 1.35$ and $\rho_c/\rho_0 = 6.0$. For the Sun this leads to a central temperature of $T_c = 6.2 \times 10^6$ K, a central density of $\rho_c = 8.4$ g/cm^3, and a central pressure of $p_c = 8.7 \times 10^9$ bar. The temperature distribution is shown in fig. 6.4 together with the data from the "standard" Sun model [11]. The agreement is reasonable, except in the deeper radiative layers and the core where it fails because we have disregarded heat production.

∗ 6.4 Field energy

What is the gravitational energy of a planet or a star? Since the gravitational potential of a finite body is always negative and grows more negative the closer one gets to the body, one does not have to perform any work to make such a body grow. It is sufficient to throw material into the general vicinity of the body, and let gravity do the rest. Consequently, the gravitational energy of a body is expected to be negative.

Gravity is in this respect different from most of the other forces we meet in daily life, for example friction, where we have to perform work to get anything done. It doesn't cost us anything to make matter collapse gravitationally, quite the contrary, we get paid for it (in heat). Matter is inherently unstable because of gravity, and this instability lies at the root of galaxy and star formation, and thus of everything that is.

Energy in external field

In chapter 3 it was shown that the work required to move a small particle of mass m from spatial infinity, where the gravitational potential vanishes, to a point $\boldsymbol{x}$, where the potential takes the value $\Phi(\boldsymbol{x})$, is $m\Phi(\boldsymbol{x})$. Consider now a body with mass density ρ in a volume V situated in an *external* potential Φ_{ext}, not originating from the mass distribution itself. The total work required to assemble the body, particle by particle brought in from spatial infinity, may then calculated by adding up the work required for each little material particle, *i.e.*

$$\boxed{W_{\text{ext}} = \int_V \Phi_{\text{ext}}\,\rho\,dV} \quad . \tag{6-31}$$

Since gravitational forces are conservative, this work is stored as potential energy of the body in the external field.

In the special case of a constant gravitational field $\boldsymbol{g}_0$ we find from (3-26)

$$W_{\text{ext}} = -\boldsymbol{x}_{\mathsf{M}} \cdot M\boldsymbol{g}_0 \ , \tag{6-32}$$

where as before $\boldsymbol{x}_M$ is the center of mass (3-3). With respect to potential energy, a body is in a constant gravitational field is also equivalent to a point particle with the total mass situated at the center of mass.

Self-energy

For a mass distribution assembled in its own field, the situation is slightly more complicated. Intuitively it is perhaps clear that each particle used to assemble the body on the average meets only half the field of the final body. Hence the energy is expected to be only half of (6-31).

To show that there is indeed such a factor 1/2 we shall employ a frequently used trick. Let us assume that a part of the mass distribution is already in place and bring in an infinitesimal extra amount of mass density $\delta\rho(\boldsymbol{x})$ to the existing distribution. The extra mass is so small that we may consider the potential of the existing masses as external and use (6-31) to compute the extra work

$$\delta W = \int_V \Phi\, \delta\rho\, dV \ . \tag{6-33}$$

Next, let us imagine that we build up the mass distribution in such a way that it is everywhere proportional to the final distribution. At any given moment, a certain fraction $\lambda\rho$ of the final distribution is already in place, where $0 < \lambda < 1$. Since the potential is linear in the mass distribution, the current potential will also be the same fraction $\lambda\Phi$ of the final potential. Increasing the fraction of the mass distribution by $\delta\lambda$ will then cost the work

$$\delta W = \int_V \lambda\Phi(\boldsymbol{r})\, \delta\lambda\, \rho(\boldsymbol{r})\, dV = \lambda\delta\lambda \int_V \Phi(\boldsymbol{r})\rho(\boldsymbol{r})\, dV \ . \tag{6-34}$$

Integrating λ from 0 to 1, we get the total amount of work we have to perform in building up the mass distribution from scratch

$$\boxed{W = \frac{1}{2}\int_V \Phi\rho\, dV} \ . \tag{6-35}$$

This work is also the total gravitational energy E_{grav} stored in a mass distribution, also called its *self-energy*.

Example 6.4.1: For a planet with constant density we get using (3-19) the gravitational self-energy

$$E_{\text{grav}} = -\frac{2}{5}\frac{GM_0^2}{a} \ , \tag{6-36}$$

when expressed in terms of the mass M_0 and the radius of the planet. For the Earth we get $E_{\text{grav}} = -1.5\times10^{32}$ J and for the Sun $E_{\text{grav}} = -1.5\times10^{41}$ J. Since the Sun's output is 3.85×10^{26} W it could only last for 3.9×10^{14} s or about 12.3 million years using gravitational energy.

Field energy density

It is possible to transform the expression for the total self-energy (6-35) into a relation involving only the field strength $\boldsymbol{g}$ by making use of the relationship

$$\boldsymbol{\nabla}\cdot(\Phi\boldsymbol{g}) = \Phi\,\boldsymbol{\nabla}\cdot\boldsymbol{g} + (\boldsymbol{g}\cdot\boldsymbol{\nabla})\Phi \ , \tag{6-37}$$

which is easily proven by writing it explicitly out in coordinates. Integrating over a volume V and using Gauss' theorem (6-3) on the left hand side we obtain

$$\oint_S \Phi\boldsymbol{g}\cdot d\boldsymbol{S} = -\int_V \boldsymbol{g}^2 - 4\pi G\int_V \Phi\rho\, dV \ ,$$

where we on the right hand side have also used eqs. (3-20) and (6-5).

If we now let the volume V expand to include not only the body but all of space, the left hand side will tend towards zero, because at large distance r we have $\Phi \sim 1/r$ and $\boldsymbol{g} \sim 1/r^2$, whereas the surface area expands only as r^2. Using this relation we may in the limit $V \to \infty$ rewrite (6-35) in the form

$$\boxed{E_{\text{grav}} = \frac{1}{2}\int \Phi\rho \, dV = -\frac{1}{8\pi G}\int \boldsymbol{g}^2 \, dV} \ , \tag{6-38}$$

where the integrals now run over all of space. This form clearly demonstrates that the energy of a self-gravitating body is negative.

In the spherical case we use (3-17) and obtain

$$E_{\text{grav}} = -\frac{1}{2}G\int_0^\infty \frac{M(r)^2}{r^2}\, dr \ . \tag{6-39}$$

This integral always converges for a body of finite mass, *i.e.* provided $M(r) \to M_0$ for $r \to \infty$, even if it has no boundary.

Where *is* the energy?

Until now we have calculated the total gravitational energy from the non-local interaction of the mass density with itself through the potential, defined by (3-24). It now seems that eq. (6-38) tells us, that it may also be viewed as arising from a local distribution of energy, the *energy density* of the gravitational field, $-\boldsymbol{g}(\boldsymbol{x})^2/8\pi G$ which is non-vanishing even in regions of space completely devoid of matter. As we discussed in section 1.4, the question whether there is *really* energy out there in space depends largely on your theoretical frame-of-mind. In classical Newtonian physics, rewriting the self-energy as an integral over an energy density is just another mathematical trick.

Problems

6.1 Show that Gauss' theorem (4-20) is a consequence of (6-3).

6.2 Show that

$$\nabla^2 \frac{1}{|\boldsymbol{x}|} = -4\pi\delta(\boldsymbol{x}) \ , \tag{6-40}$$

where $\delta(\boldsymbol{x})$ is the three-dimensional δ-function, *i.e.* the mass distribution of a unit mass point particle at the origin.

6.3 The gravitational field of a fluid itself influences the pressure in it. Show that for the flat Earth, where all quantities depend on z only, the hydrostatic equilibrium equation takes the form

$$\frac{d}{dz}\left(\frac{1}{\rho}\frac{dp}{dz}\right) = -4\pi G\rho \tag{6-41}$$

a) Calculate how the pressure in the incompressible sea is changed by its own gravity.
b) Do the same for the isentropic atmosphere.

∗ **6.4** Show that the "inverse" of the Laplace operator is the integral operator

$$(\nabla^2)^{-1}(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{4\pi\,|\boldsymbol{x} - \boldsymbol{x}'|} \tag{6-42}$$

To which extent is this inverse unique?

6.5 Show that the equation of hydrostatic equilibrium (6-12) may be rewritten in the form

$$\frac{1}{\rho}\nabla^2 p - \frac{1}{\rho^2}\frac{d\rho}{dp}(\nabla p)^2 = -4\pi G\rho \tag{6-43}$$

6.6 Show that in a spherical layer (of a spherical planet) with constant density ρ_0 , the quantity $p(r) + \rho_0\Phi(r)$ is a constant.

6.7 Calculate the hydrostatic pressure in a two-layer planet (see problem 3.11).

6.8 Show that for a planet with constant density and fixed mass, the central pressure falls like a^{-4}.

6.9 Find the pressure in a planet made of a material with constant compressibility.

6.10 Show that the adiabatic index for an ideal gas in isentropic equilibrium is given by

$$1 - \frac{1}{\gamma} = \frac{d\log T}{d\log p} \tag{6-44}$$

6.11 a) Find the power law solutions to the stellar equation (6-21) of the form $T \sim r^\alpha$ with $\alpha < 0$. b) Determine the condition for finite mass for $r \to 0$. c) Determine the condition for finite energy (6-39) for $r \to 0$.

6.12 Show that the short distance behaviour of the Lane-Emden functions is $\theta(s) = 1 - s^2/6$, independently of γ.

6.13 a) Show that for $\gamma = 6/5$ the solution to the Lane-Emden equation is $\theta(s) = (1 + s^2/3)^{-1/2}$. b) Calculate pressure and density. c) Show that although the star has no boundary, it nevertheless has finite mass.

6.14 Compare the gravitational energy of the Earth to an estimate of how much energy would be needed to melt the Earth. Do you think the Earth melted when its material was accumulated from an early cold cloud around the Sun?

6.15 Compare for a spherical planet with constant mass density the total field energy inside the planet with the field energy outside. For a black hole with $a = GM_0/c^2$ and total energy M_0c^2, calculate the fraction of the total mass which is due to the field.