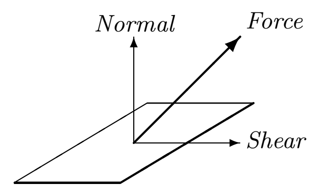
Part III

Deformable solids

8

Stress

In a fluid at rest, pressure is the only contact force. For solids at rest or in motion, and for viscous fluids in motion, this simple picture is no longer valid. Besides pressure-like forces acting along the normal to a contact surface, there may also be shear forces acting tangentially to it. In complete analogy with pressure, the relevant quantity turns out to be the *shear stress*, defined to be the shear force per unit of area. Friction forces are always due to shear stresses.



The force on a small piece of a surface can be resolved in a normal pressure-like force and a tangential shear force.

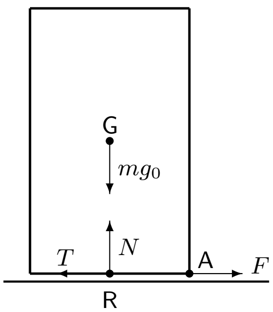
The two major classes of materials, fluids and solids, react differently to stress. Whereas fluids respond by *flowing*, solids respond by *deforming*. Although the equations of motion in both cases are derived from Newton's second law, fluids and solids are in fact so different, that they usually are covered in separate textbooks. In this book, we shall as far as possible maintain a general view of the physics of continuous systems, applicable to all types of materials.

The integrity of a solid body is largely due to internal elastic stresses, both normal and shear. Together they resist deformation of the material and prevent the body from being pulled apart. Unlike friction, elastic forces do not dissipate energy, and ideally the work done against elastic forces during deformation may be fully recovered. In reality, some elastic energy will always be lost because of emission of sound waves that eventually decay and turn into heat.

Baron Augustin Louis Cauchy (1789–1857). *French mathematician. Contributed to the foundations of elasticity, hydrodynamics, and complex analysis*

In this chapter the emphasis is on the theoretical formalism for contact forces, independently of whether they occur in solids, fluids, or intermediate plastic forms such as clay or dough. The vector notation used up to this point is not adequate to the task, because contact forces not only depend on the spatial position but also on the orientation of the surface on which they act. A collection of nine stress components, called the *stress tensor*, was introduced by Cauchy in 1822 to describe the full range of contact forces that may come into play in a body.

8.1 Friction

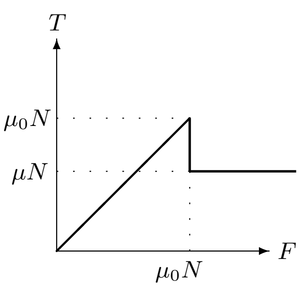


Balance of forces on a crate standing still on a horizontal floor. The normal reaction N balances gravity mg_0 and the tangential reaction T balances the external force F. The point of attack A for the external force is here chosen to be at floor level to avoid creating a moment of force which could turn over the crate.

The concept of shear stress is best understood through *friction*, a shear force known to us all. We hardly think of friction forces, even though we all day long are served by them and do service to them. Friction is the reason that the objects we hold are not slippery as a piece of soap in the bathtub, but instead allow us to grab and drag, heave and lift, rub and scrub. Most of the work we do is in fact done against friction, from stirring the coffee to making fire by rubbing two sticks against each other.

Static and sliding friction

Consider a heavy crate standing on a horizontal floor. Its weight mg_0 acts vertically downwards on the floor, which in turn reacts back on the crate with an equal and opposite normal force of magnitude $N = mg_0$. If you try to drag the crate along the floor by applying a horizontal force F, also called *traction*, you may discover that the crate is so heavy that you are not able to budge it, implying that the force that you are able to apply must be fully balanced by a tangential friction force between the floor and the crate of the same magnitude, $T = F$, but of opposite direction.



Sketch of tangential reaction T as a function of applied traction F. Up to $F = \mu_0 N$, the tangential reaction adjusts itself to the traction, $T = F$. At $F = \mu_0 N$, the tangential reaction drops abruptly to a lower value, and stays there independently of the applied traction. Since the forces are no more in balance the body has to move.

Empirically, such *static friction* can take any magnitude up to a certain maximum, which is proportional to the normal load,

$$T < \mu_0 N \; . \tag{8-1}$$

The dimensionless constant of proportionality μ_0 is called the coefficient of static friction which in our daily doings may take a quite sizable value, say 0.5 or greater. Its value depends on what materials are in contact and on the roughness of the contact surfaces.

If you are able to pull with a sufficient strength, the crate suddenly starts to move, but friction will still be present and you will have to do real work to move the crate any distance. Empirically, the *dynamic* (kinetic or sliding) friction is proportional to the normal load,

$$T = \mu N \; , \tag{8-2}$$

with a coefficient of dynamic friction, μ, always smaller than the corresponding coefficient of static friction, $\mu < \mu_0$ This is why you have to heave strongly to get the crate set into motion, whereas afterwards a smaller force suffices to keep it going at constant speed. The law of sliding friction goes back to Coulomb (1785) (and Amontons (1699)). The full story of dynamic friction is complicated, and in spite of the everyday familiarity with friction, there is still no universally accepted microscopic explanation of the phenomenon [43].

Charles-Augustin de Coulomb (1736–1806). *French physicist best known from the electrostatic law that carries his name.*

Example 8.1.1: The proportionality between friction and load is also the reason that a car's braking distance in the leading approximation is independent of how heavily it is loaded. Anti-skid brake systems automatically adjust braking pressure so that you can avoid skidding and thus all the time exploit that static friction is greater than sliding friction in order to minimize braking distance.

Stress and friction

Shear stress is, just like pressure, defined as force per unit of area, and the standard unit of stress is the same as the unit for pressure, namely pascal ($\mathsf{Pa} = \mathsf{N/m^2}$). If the crate on the floor has a contact area A, we may speak both about the average normal stress $\sigma_n = N/A$ and the average tangential (or shear) stress $\sigma_t = T/A$ that the crate exerts on the floor. In terms of the stresses, the laws of static and dynamic friction take the form

$$\sigma_t < \mu_0\,\sigma_n\ , \qquad\qquad \sigma_t = \mu\,\sigma_n\ . \tag{8-3}$$

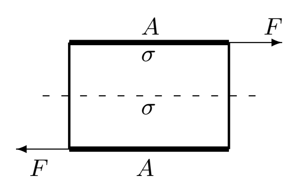
Depending on the mass distribution of the contents of crate and the stiffness of its bottom, the local stresses may vary across the contact area A. For a planar contact area the total normal and tangential reactions become integrals over the normal and tangential local stresses,

$$N = \int_A \sigma_n(\boldsymbol{x})\,dS\ , \qquad\qquad T = \int_A \sigma_t(\boldsymbol{x})\,dS\ . \tag{8-4}$$

If the stress distributions everywhere obey the local friction laws (8-3), the global friction laws (8-1) and (8-2) follow automatically.

8.2 Internal stresses

The stresses acting between the crate and the floor are *external* and are found in the true interface between a body and its environment. In analogy with pressure, we shall also speak about *internal* stresses, even if we may be unable to define a practical way to measure them. Internal stresses abound in the macroscopic world around us. Whenever we come into contact with the environment (and when do we not) stresses are set up in the materials we touch, and in our own bodies. The precise distribution of stress in a body depends not only on the external forces applied to the body, but also on the type of material the body is made from and on other macroscopic quantities such as temperature. In the absence of external forces there is usually no stress in a material, although fast cooling may freeze stresses permanently into certain materials, for example glass, and provoke an almost explosive release of stored energy when triggered by a sudden impact.

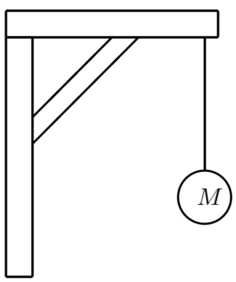


Clamped slab of homogeneous material. The shear force F at the upper clamp is balanced by an oppositely directed fixation force F on the lower clamp. The shear stress $\sigma = F/A$ is everywhere the same on all inner surfaces parallel with the clamps.

Estimating internal stress

In many situations it is quite straightforward to estimate average stresses in a body. Consider, for example, a slab of homogeneous solid material bounded by two stiff flat clamps of area A, firmly glued to it. A tangential force of magnitude F applied to one clamp with the other held fixed will deform the slab a bit in the direction of the applied force. Here we shall not worry about how to calculate the deformation of the slab, but just assume that the response of the slab is the same everywhere, so that there is a uniform shear stress $\sigma = F/A$ acting on the surface of the slab.

The fixed clamp will of course act back on the slab with a force of the same magnitude but opposite direction. If we make an imaginary cut through the slab parallel with the clamps, then the upper part of the slab must likewise act on the lower with the shear force F, so that the internal shear stress everywhere in the cut again must be $\sigma = F/A$. If pressure had also been applied to the clamps, we would have gone through the same type of argument to convince ourselves that the normal stress would be the same everywhere in the cut.



The classic gallows.

For bodies with a more complicated geometry and non-uniform external load, internal stresses are not so easily calculated, although their average magnitudes may be estimated. In analogy with friction, for many non-exceptional materials and body geometries one may assume that variations in shear and normal stresses are roughly of the same order of magnitude.

Example 8.2.1: The classic gallows is constructed from a vertical pole, a horizontal beam, and sometimes a diagonal strut. A body of mass $M = 70$ kg hangs at the extreme end of the horizontal beam, of cross-section $A = 100$ cm^2. The body's weight must be balanced by a vertical shear stress in the beam of magnitude $\sigma \approx Mg_0/A \approx 70{,}000$ Pa, or 0.7 bar. The actual distribution of shear stress will vary over the cross-section of the beam and the position of the chosen cross-section, but its average magnitude should be of the estimated value.

Example 8.2.2: The half-inch water mains in your house have an inner pipe radius $a \approx 0.6$ cm. Tapping water at a high rate, internal friction in the water (viscosity) creates shear stresses opposing the flow, and the pressure drops perhaps by $\Delta p \approx 0.1$ bar $= 10^4$ Pa over a length of $L \approx 10$ m of the pipe. In this case, we may actually calculate the shear stress on the water from the inner surface of the pipe without estimation errors, because the pressure difference between the ends of the pipe is the only other force acting on the water. Setting the force due to the pressure difference equal to the total shear force on the inner surface, we get, $\pi a^2 \Delta p = 2\pi a L \sigma$, from which it follows that $\sigma = \Delta p\, a/2L \approx 3$ Pa. This stress is indeed of the same size as we would have estimated from the corresponding pressure drop $\Delta p \cdot a/L$ over a stretch of pipe of the same length as the radius.

Metal	*MPa*
Lead	*17*
Zink	*130*
Cast iron	*180*
Copper	*300*
Titanium	*330–500*
Carbon steel	*450*
Nickel	*460*
Stainless steel	*550*

Typical tensile strength for common metals. The values may vary widely for different specimens, depending on heat treatment and other factors.

Tensile strength

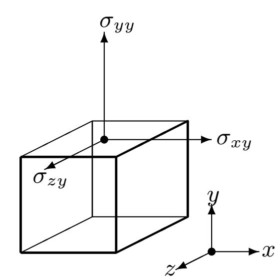
When external forces grow large, a solid body may fracture and break apart. The maximal tension, *i.e.* negative pressure or pull, a material can sustain without fracturing is called the *tensile strength* of the material. For metals it is typically in the region of hundreds of megapascals.

Example 8.2.3: Plain carbon steel has a tensile strength of 450 MPa. A quick estimate shows that a steel rod with a diameter of 2 cm breaks, if loaded with more than 14,000 kg. Adopting a safety factor of 10, one should not load it with more than 1,400 kg.

The *yield stress* is defined as the stress beyond which otherwise elastic solids begin to undergo permanent deformation.

8.3 Nine components of stress

Shear stress is more complicated than normal stress, because there is more than one tangential direction on a surface. In a coordinate system where a force $d\mathcal{F}_x$ is applied along the x-direction to material surface dS_y with its normal in the y-direction, the shear stress will be denoted $\sigma_{xy} = d\mathcal{F}_x/dS_y$, instead of just σ. Similarly, if the shear force is applied in the z-direction, the stress would be denoted $\sigma_{zy} = d\mathcal{F}_z/dS_y$, and if a normal force had been applied along the y-direction, it would be consistent to denote the normal stress $\sigma_{yy} = d\mathcal{F}_y/dS_y$. By convention, the sign is chosen such that a positive value of σ_{yy} corresponds to a pull or *tension*.



Components of stress acting on a surface element in the xz-plane.

Cauchy's stress hypothesis

Altogether, it therefore appears to be necessary to use at least nine numbers to indicate the state of stress in a given point of a material in a particular coordinate system. *Cauchy's stress hypothesis* (to be proved below) asserts that the force $d\boldsymbol{\mathcal{F}} = (\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_z)$ on an arbitrary surface element, $d\boldsymbol{S} = (dS_x, dS_y, dS_z)$, with arbitrary orientation with respect to the coordinate frame, is of the form

$$\begin{aligned} d\mathcal{F}_x &= \sigma_{xx} dS_x + \sigma_{xy} dS_y + \sigma_{xz} dS_z \ , \\ d\mathcal{F}_y &= \sigma_{yx} dS_x + \sigma_{yy} dS_y + \sigma_{yz} dS_z \ , \\ d\mathcal{F}_z &= \sigma_{zx} dS_x + \sigma_{zy} dS_y + \sigma_{zz} dS_z \ . \end{aligned} \tag{8-5}$$

where each coefficient $\sigma_{ij} = \sigma_{ij}(\boldsymbol{x}, t)$ depends on the position and time, and thus is a field in the normal sense of the word. Collecting them in a matrix

$$\boldsymbol{\sigma} = \{\sigma_{ij}\} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \ . \tag{8-6}$$

the force may be written compactly,

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{\sigma} \cdot d\boldsymbol{S}} \ . \tag{8-7}$$

The force per unit of area, $d\boldsymbol{\mathcal{F}}/dS = \boldsymbol{\sigma} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the normal to the surface, is sometimes called the *stress vector*, although it is *not a vector field* in the usual sense of the word because it depends on the normal.

The stress tensor

Together the nine fields, $\{\sigma_{ij}\}$, make up a single geometric object, called the *stress tensor*, first introduced by Cauchy in 1822. Using index notation, we may write

$$\boxed{d\mathcal{F}_i = \sum_j \sigma_{ij} dS_j} \quad . \tag{8-8}$$

Since the force $d\mathcal{F}_i$ as well as the surface element dS_i are vectors, it follows that σ_{ij} is indeed a tensor in the sense of in section 2.6 (see also problem 2.17). This collection of nine fields $\{\sigma_{ij}\}$ cannot be viewed geometrically as consisting of nine scalar or three vector fields, but must be considered together as one geometrical object, a *tensor field* $\sigma_{ij}(\boldsymbol{x}, t)$ which is neither scalar nor vector. As for ordinary tensors (see section 2.6), there is unfortunately no simple, intuitive way of visualizing the stress tensor geometrically.

Example 8.3.1: The stress tensor field of the form,

$$\{\sigma_{ij}\} = \{x_i x_j\} = \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix} \tag{8-9}$$

is a tensor product and thus by construction a true tensor. The stress "vector" acting on a surface with normal in the direction of the x-axis is

$$\boldsymbol{\sigma}_x = \boldsymbol{\sigma} \cdot \boldsymbol{e}_x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} x \tag{8-10}$$

does not transform under rotations as a true vector because of the factor x on the right hand side.

Hydrostatic pressure

For the special case of hydrostatic equilibrium, where the only contact force is pressure, comparison of (8-7) with (4-7) shows that the stress tensor must be

$$\boldsymbol{\sigma} = -p\, \mathbf{1} \ , \tag{8-11}$$

where $\mathbf{1}$ is the $[3 \times 3]$ unit matrix. In tensor notation this becomes

$$\sigma_{ij} = -p\, \delta_{ij} \ , \tag{8-12}$$

where δ_{ij} is the index representation of the unit matrix, *i.e.* the Kronecker delta (2-27).

Average pressure

Generally, however, the stress tensor will have both diagonal and off-diagonal non-vanishing components. A diagonal component behaves like a (negative) pressure, and one often defines the pressures along different coordinate axes to be

$$p_x = -\sigma_{xx} \ , \quad p_y = -\sigma_{yy} \ , \quad p_z = -\sigma_{zz} \ . \tag{8-13}$$

Since they may be different, it is not clear what the meaning of *the* pressure in a point should be. Furthermore, it should be remembered that the diagonal elements of a tensor $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ do not behave as a vector under Cartesian coordinate transformations and thus have no well-defined geometric meaning (see section 2.6 and problem 2.21).

The pressure is defined to be the average of the three pressures along the axes,

$$p = \frac{1}{3}(p_x + p_y + p_z) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \ . \tag{8-14}$$

This makes sense because the sum over the diagonal elements of a matrix, the trace Tr $\boldsymbol{\sigma} = \sum_i \sigma_{ii} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$, is invariant under Cartesian coordinate transformations (problem 2.15). Defining pressure in this way ensures that it is a scalar field, taking the same value in all coordinate systems.

Example 8.3.2: For the stress tensor given in example 8.3.1 the pressures along the coordinate axes become

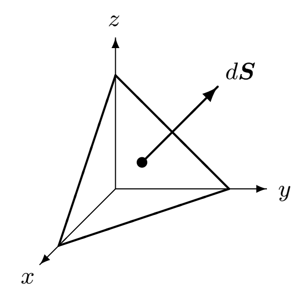
$$p_x = -x^2 \ , \quad p_y = -y^2 \ , \quad p_z = -z^2 \ . \tag{8-15}$$

The average pressure,

$$p = -\frac{1}{3}(x^2 + y^2 + z^2) \ , \tag{8-16}$$

proportional to the distance squared from the origin, is clearly invariant under rotations of the Cartesian coordinate system.

* Proof of Cauchy's stress hypothesis



The tiny triangle and its projections form a tetrahedron.

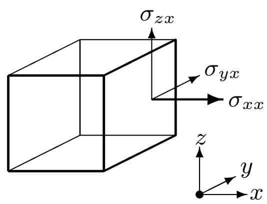
Let us as in the proof of Pascal's law (page 64) again take a surface element in the shape of a tiny triangle with area vector $d\boldsymbol{S} = (dS_x, dS_y, dS_z)$. The triangle and its projections on the coordinate planes form together a little body in the shape of a tetrahedron. Since we aim to prove the existence of the stress tensor, we cannot assume that it exists. What we know is that the forces acting from the inside of the tetrahedron on the three triangular faces in the coordinate planes are vectors of the form $d\boldsymbol{\mathcal{F}}_x = \boldsymbol{\sigma}_x dS_x$, $d\boldsymbol{\mathcal{F}}_y = \boldsymbol{\sigma}_y dS_y$, and $d\boldsymbol{\mathcal{F}}_z = \boldsymbol{\sigma}_z dS_z$. Calling the force acting from the outside on the fourth (skew) face $d\boldsymbol{\mathcal{F}}$, and adding a possible volume force $\boldsymbol{f} dV$, the equation of motion for the small tetrahedron becomes

$$dM\,\boldsymbol{w} = \boldsymbol{f} dV + d\boldsymbol{\mathcal{F}} - d\boldsymbol{\mathcal{F}}_x - d\boldsymbol{\mathcal{F}}_y - d\boldsymbol{\mathcal{F}}_z \ , \tag{8-17}$$

where $\boldsymbol{w}$ is the acceleration of the tetrahedron, and $dM = \rho dV$ its mass, which is assumed to be constant. The signs have been chosen in accordance with the inward direction of the area projections dS_x, dS_y and dS_z.

The volume of the tetrahedron scales like the third power of its linear size, whereas the surface areas only scale like the second power (see section 4.1). Making the tetrahedron progressively smaller, the body force term and the left hand side of the above equation will vanish faster than the surface terms. In the limit of a truly infinitesimal tetrahedron, only the surface terms survive, so that we must have

$$d\boldsymbol{\mathcal{F}} = \boldsymbol{\sigma}_x dS_x + \boldsymbol{\sigma}_y dS_y + \boldsymbol{\sigma}_z dS_z \ . \tag{8-18}$$



Components of the stress vector $\boldsymbol{\sigma}_x$ acting on a surface element in the yz-plane.

This shows that the force on an arbitrary surface element may be written as a linear combination of three basic stress vectors, one for each coordinate axis. Introducing the nine coordinates of these three vectors, $\boldsymbol{\sigma}_x = (\sigma_{xx}, \sigma_{yx}, \sigma_{zx})$, $\boldsymbol{\sigma}_y = (\sigma_{xy}, \sigma_{yy}, \sigma_{zy})$, and $\boldsymbol{\sigma}_z = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$, we arrive at (8-5).

8.4 Mechanical equilibrium

Including a volume force density f_i, the total force on a volume V with surface S becomes according to (8-8)

$$\mathcal{F}_i = \int_V f_i \, dV + \oint_S \sum_j \sigma_{ij} \, dS_j \ . \tag{8-19}$$

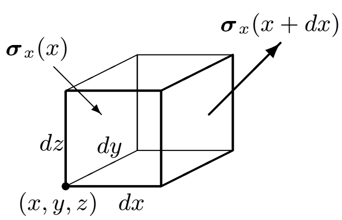
Using Gauss' theorem (4-20) this may be written as single volume integral

$$\mathcal{F}_i = \int_V f_i^* \, dV \ , \tag{8-20}$$

where

$$f_i^* = f_i + \sum_j \nabla_j \sigma_{ij} \ , \tag{8-21}$$

is called the *effective force density*. The effective force is not just a formal quantity, because the total force on a material particle of volume dV is $d\boldsymbol{\mathcal{F}} = \boldsymbol{f}^* \, dV$. This may as in hydrostatics also be shown by considering a small box-shaped rectangular particle.



The total contact force on a small box-shaped material particle is calculated from the variations in stress on the sides. Thus $d\boldsymbol{\mathcal{F}} = (\boldsymbol{\sigma}_x(x + dx, y, z) - \boldsymbol{\sigma}_x(x, y, z))dS_x \approx \nabla_x \boldsymbol{\sigma}_x \, dV$ for the stress on dS_x, plus the similar contributions from dS_y and dS_z.

Cauchy's local equilibrium equation

In mechanical equilibrium, the total force on any piece of material must vanish, for if it doesn't the piece of material will begin to move. So the general condition is that $\boldsymbol{\mathcal{F}} = \boldsymbol{0}$ for all volumes V. As for hydrostatic equilibrium, it is advantageous to formulate the principle of mechanical equilibrium for infinitesimal material particles, thereby liberating the formalism for the explicit volume of matter V.

In complete mechanical equilibrium the effective force density has to vanish everywhere, leading to the partial differential equations

$$\boxed{f_i + \sum_j \nabla_j \sigma_{ij} = 0 \ .} \tag{8-22}$$

This equation is called *Cauchy's equation of equilibrium* (1827) and governs in spite of its apparent simplicity mechanical equilibrium in all kinds of continuous matter, be it solid, fluid, or something else. In particular, for $\sigma_{ij} = -p\,\delta_{ij}$ we recover the equation of hydrostatic equilibrium, $f_i - \nabla_i p = 0$.

The three individual equations in Cauchy's equilibrium equation,

$$\begin{aligned} f_x + \nabla_x \sigma_{xx} + \nabla_y \sigma_{xy} + \nabla_z \sigma_{xz} = 0 \\ f_y + \nabla_x \sigma_{yx} + \nabla_y \sigma_{yy} + \nabla_z \sigma_{yz} = 0 \\ f_z + \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} + \nabla_z \sigma_{zz} = 0 \end{aligned} \tag{8-23}$$

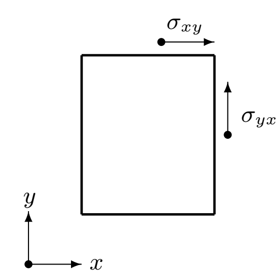
are insufficient in number to determine the equilibrium, and must be supplemented by suitable *constitutive* equations connecting stress with the state of matter. For fluids at rest, the equation of state serves this purpose by relating hydrostatic pressure and mass density. In elastic solids, the constitutive equations are more complicated and relate stress to deformation (chapter 10).

Fluids and solids in motion can by their nature not be in mechanical equilibrium and obey instead dynamic equations that we shall discuss in chapters 14 and 12. In addition to hydrostatic pressure, fluids in motion will also be subject to stresses that depend on the spatial variation in flow velocity (chapter 17).

Symmetry

There is one very general condition (also going back to Cauchy) which may always be imposed, namely the symmetry of the stress tensor

$$\boxed{\sigma_{ij} = \sigma_{ji} \ .} \tag{8-24}$$



A symmetric stress tensor acts with equal strength on orthogonal faces of a cubic body.

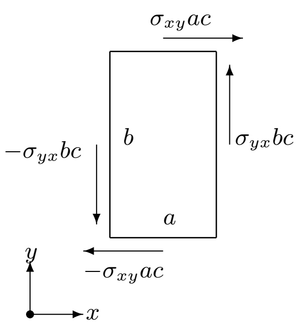
Symmetry only affects the shear stress components, requiring

$$\sigma_{xy} = \sigma_{yx} \ , \qquad \sigma_{yz} = \sigma_{zy} \ , \qquad \sigma_{zx} = \sigma_{xz} \ , \tag{8-25}$$

and thus reduces the number of independent stress components from nine to six.

Being thus a symmetric matrix, the stress tensor may be diagonalized. The eigenvectors define the principal directions of stress and the eigenvalues the principal tensions or stresses. In the principal basis, there are no off-diagonal elements, *i.e.* shear stresses, only pressures. The principal basis is generally different from point to point.

Proof of symmetry: The technical "proof" of symmetry rests on exploiting an ambiguity in the definition of the stress tensor and will be given in section 8.5. Here we shall only present a simple argument valid for mechanical equilibrium. Consider again a material particle in the shape of a tiny rectangular box with sides a, b, and c. The force acting in the y-direction on a face in the x-direction is $\sigma_{yx}bc$ whereas the force acting in the x-direction on a face in the y-direction is $\sigma_{xy}ac$. On the opposite faces the contact forces have opposite sign in mechanical equilibrium (their difference and the volume forces are as we have seen of order abc). The total moment of force on the box then becomes (calculated around the lower left corner)

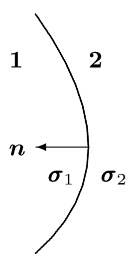


An asymmetric stress tensor will produce a non-vanishing moment of force on a small box (the z-direction not shown).

$$\mathcal{M}_z = a\,\sigma_{yx}bc - b\,\sigma_{xy}ac = (\sigma_{yx} - \sigma_{xy})abc\ .$$

This shows that if the stress tensor is asymmetric, $\sigma_{xy} \neq \sigma_{yx}$, there will be a resultant moment on the box. In mechanical equilibrium this cannot be allowed, since such a moment would begin to rotate the box, and consequently the stress tensor must be symmetric. Conversely, when the stress tensor is symmetric, mechanical equilibrium of the forces alone guarantees that all moments of force will vanish.

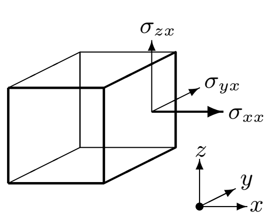
Boundary conditions



Contact surface separating body 1 from body 2. Newton's third law requires continuity of the stress vector $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ across the boundary, i.e. $\boldsymbol{\sigma}_1\cdot\boldsymbol{n} = \boldsymbol{\sigma}_2\cdot\boldsymbol{n}$.

Cauchy's equation of equilibrium is a differential equation, and differential equations require boundary conditions. The stress tensor is a local physical quantity, or rather collection of quantities, and may, like pressure in hydrostatics, be assumed to be continuous in regions where material properties change continuously. Across real boundaries, interfaces, where material properties may change abruptly, Newton's third law only demands that the stress vector, $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \{\sum_j \sigma_{ij}n_j\}$, be continuous across a surface with normal $\boldsymbol{n}$. This does *not* mean that all the components of the stress tensor should be continuous. Since Newton's third law is a vector condition, it imposes continuity on three linear combinations of stress components, but leaves for the symmetric stress tensor three other combinations, among these the average pressure, free to jump discontinuously.

Example 8.4.1: Consider a plane interface in the yz-plane. The stress components σ_{xx}, σ_{yx}, and σ_{zx} must then be continuous, because they specify the stress vector on such a surface. Symmetry implies that σ_{xy} and σ_{xz} are likewise continuous. The remaining three independent components σ_{yy}, σ_{zz}, and $\sigma_{yz} = \sigma_{zy}$ are allowed to jump at the interface. In particular the average pressure, $p = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$, may jump.



Only the three components of the stress vector need to be continuous on the interface.

* 8.5 “Proof” of symmetry of the stress tensor

If the stress tensor is manifestly *asymmetric*, we shall now show that it is always possible to make it symmetric by exploiting an ambiguity in its definition. The argument which will now be presented is adapted from Martin, Parodi, and Pershan[1] (see also [10, p. 7]).

The stress tensor was introduced in the beginning of this chapter as a quantity which furnished a complete description of the contact forces that may act on any surface element. But surface elements are not in themselves physical bodies. The only way we can determine the magnitude and direction of a force is by observing its influence on the motion of a real physical body having a volume and a closed surface. The resultant of all contact forces acting on the surface of a body is

$$\oint_S \sum_j \sigma_{ij}\, dS_j = \int_V \sum_j \nabla_j \sigma_{ij}\, dV \ ,$$

and this shows that the relevant quantity for the dynamics of continuous matter is the effective density of force $\sum_j \nabla_j \sigma_{ij}$ rather than the stress tensor itself.

Two stress tensors, σ_{ij} and $\widetilde{\sigma}_{ij}$, are therefore physically indistinguishable, if they give rise to the same effective density of force everywhere. This is, for example, the case if we write

$$\widetilde{\sigma}_{ij} = \sigma_{ij} + \sum_k \nabla_k \chi_{ijk} \tag{8-26}$$

where χ_{ijk} is antisymmetric in j and k,

$$\chi_{ijk} = -\chi_{ikj} \ . \tag{8-27}$$

For then

$$\sum_j \nabla_j \widetilde{\sigma}_{ij} = \sum_j \nabla_j \sigma_{ij} + \sum_{jk} \nabla_j \nabla_k \chi_{ijk} = \sum_j \nabla_j \sigma_{ij} \ ,$$

where the last term in the middle vanishes because of the symmetry of the double derivatives and the assumed antisymmetry of χ_{ijk}.

It remains to show that there exists a tensor χ_{ijk} such that $\widetilde{\sigma}_{ij}$ becomes symmetric. Let us put

$$\chi_{ijk} = \nabla_i \phi_{jk} + \nabla_j \phi_{ik} - \nabla_k \phi_{ij} \tag{8-28}$$

where ϕ_{ij} is an antisymmetric tensor, $\phi_{ij} = -\phi_{ji}$, chosen to be a solution to Poisson's equation with the antisymmetric part of the original stress tensor as source,

$$\boldsymbol{\nabla}^2 \phi_{ij} = \frac{1}{2}\left(\sigma_{ij} - \sigma_{ji}\right) \ . \tag{8-29}$$

[1] P. C. Martin, O. Parodi, and P. S. Pershan, Phys. Rev. **A6**, 2401 (1972)

Such a solution can in principle always be found, and then we obtain from (8-26)

$$\widetilde{\sigma}_{ij} = \frac{1}{2}\bigl(\sigma_{ij} + \sigma_{ji}\bigr) + \sum_k \nabla_k\bigl(\nabla_i\phi_{jk} + \nabla_j\phi_{ik}\bigr) \tag{8-30}$$

which is manifestly symmetric. Notice, however, that the new symmetric stress tensor is not just the symmetric part of the old, but contains extra terms.

Non-classical continuum theories

The conclusion is, that if somebody presents you with a stress tensor which is asymmetric, you may always replace it by a suitable symmetric stress tensor, having exactly the same physical consequences.

But even if it is formally possible to choose a symmetric stress tensor, it may not always be convenient, because of the non-locality inherent in the solution to Poisson's equation in (8-29). Asymmetric stress tensors have been used in various generalizations of classical continuum theory, containing elementary volume and surface densities of moments (body couples and couple stresses) and sometimes also intrinsic angular momentum (spin). We shall not go further into these extensions of continuum theory here (so-called *micropolar* materials are, for example, discussed in [33, p. 493]).

Problems

8.1 A crate standing on a horizontal floor is pulled with a force F attacking at a height h above the floor. Show that the vanishing of the moment of force implies that the normal reaction force N must attack in a point which is positioned a horizontal distance d from the center of gravity.

Determine the angle α with the vertical of the total reaction force for a crate being dragged over a horizontal floor with sliding friction coefficient μ.

8.2 A car with mass m moves with a speed v. Estimate the minimal breaking distance without skidding and the corresponding braking time. Do the same if it skids from the beginning to the end. For numerics use $m = 1000$ kg and $v = 100$ km/h. The static coefficient of friction between rubber and the surface of a road may be taken to be $\mu_0 = 0.5$, whereas the sliding friction is $\mu = 0.4$.

8.3 A body of mass m stands still on a horizontal floor. The coefficients of static and kinetic friction between body and floor are μ_0 and μ. An elastic string with string constant k is attached to the body in a point close to the floor. The string can only exert a force on the body when it is stretched beyond its relaxed length. When the free end of the string is pulled horizontally with constant velocity v, intuition tells us that the body will have a tendency to move in fits and starts.

a) Calculate the amount s that the string is stretched, just before the body begins to move?

b) Write down the equation of motion for the body when it is just set into motion, for example in terms of the distance x that the point of attachment of the string has moved and the time t elapsed since the motion began.

c) Show that the solution to this equation is

$$x = \frac{v}{\omega}(\omega t - \sin \omega t) + (1 - r)s(1 - \cos \omega t)$$

where $\omega = \sqrt{k/m}$, $r = \mu/\mu_0$.

d) Assuming that the string stays stretched, calculate at what time $t = t_0$ the body stops again?

e) Find the condition for the string to be stretched during the whole motion.

f) How long time will the body stay in rest, before moving again?

8.4 A strong man pulls a jumbo airplane slowly but steadily exerting a force of 2000 N on a rope. The plane has 32 wheels each touching the ground in a square area with side 40 cm. a) Estimate the shear stress due to friction between the rubber and the tarmac. b) Estimate the shear stress between the tarmac and his feet of size 10×25 cm^2.

8.5 Estimate the maximal height of a mountain made from rock with a density of $3,000$ kg/m^3 when the maximal stress the material can tolerate before it deforms permanently is 1000 MPa.

8.6 Buy a stick of rubbery candy and estimate (or measure) its tensile strength.

8.7 Show that if the stress tensor is diagonal in all coordinate systems, then it can only contain pressure.

8.8 A stress tensor has all components equal, *i.e.* $\sigma_{ij} = \tau$ for all i, j. Find its eigenvalues and eigenvectors.

8.9 Show without using Gauss' theorem that the sum of all mechanical forces on an infinitesimal rectangular box of volume $dV = dxdydz$ is $\boldsymbol{f}^* \, dV$.

∗ **8.10** Show that the minimum and maximum of $\sigma_{jk} n_j n_k$, where $\boldsymbol{n}$ is a unit vector, occurs along the principal axes.

∗ **8.11** One may define three *invariants*, *i.e.* scalar functions, of the stress tensor in any point. The first is the trace $I_1 = \sum_i \sigma_{ii}$, the second $I_2 = \frac{1}{2} \sum_{ij} (\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})$ which has no special name, and the determinant $I_3 = \det \boldsymbol{\sigma}$. Write the characteristic equation for the matrix $\boldsymbol{\sigma}$ in terms of the invariants. Can you find a further invariant for an asymmetric stress tensor?