18.5 One verifies explicitly that the expression satisfies the equation of motion (18-5). The constant in front is determined by requiring

$$\int_{-\infty}^{\infty} v_x(y,t)\,dy = \int_{-\infty}^{\infty} v_x(y,0)\,dy$$

For $t \to 0$ the Gaussian becomes infinitely narrow (a δ-function) and thus $v_x(y,t) \to v_x(y,0)$. Finally, assuming that $v_x(y,0) = 0$ for $|y| \leq a$ one gets for $|y| \to \infty$ and $4\nu t \gg a^2$

$$v_x(y,t) \approx \frac{1}{2\sqrt{\pi\nu t}} \exp\left(-\frac{y^2}{4\nu t}\right) \int_{-\infty}^{\infty} v_x(y',0)\,dy' \qquad \text{(18-A7)}$$

18.6 a) The average of $\langle n_i n_j \rangle$ over all directions of $\boldsymbol{n}$ does not itself depend on any direction, so that it must be proportional to Kronecker's delta, $\langle n_i n_j \rangle = k\delta_{ij}$. The constant k is determined by taking the trace of both sides, $1 = \langle \boldsymbol{n}^2 \rangle = 3k$.

18.7 a) $L \approx 10$ km, $U \approx 1$ m/s, $\mathrm{Re} \approx 10^{10}$. b) $L \approx 30$ m, $U \approx 30$ m/s, $\mathrm{Re} \approx \times 10^9$. c) $L \approx 1000$ km, $U \approx 10$ m/s, $\mathrm{Re} \approx 10^{12}$. d) $L \approx 500$ km, $U \approx 50$ m/s, $\mathrm{Re} \approx 3 \times 10^{12}$. e) $L \approx 1$ km, $U \approx 100$ m/s, $\mathrm{Re} \approx 10^{10}$

19 Plates and tubes

19.1 Use the general solution (19-7) and the no-slip conditions to get

$$v_x = \frac{G}{2\eta} y(d-y) + U\frac{y}{d}$$

The maximum happens at $y = \dfrac{d}{2} + \dfrac{U\eta}{Gd}$ and lies between the plates for $2U\eta < Gd^2$.

19.2 Let the pressure gradient be G along the x-direction and the relative plate velocity U along the z-direction. Assume that the field is of the form $\boldsymbol{v} = (v_x(y), 0, v_z(y))$. Then the Navier-Stokes equations imply that p and v_x are as in the planar pressure driven case, whereas v_z is as in the velocity driven case.

19.3 If pressure were used to drive the planar sheet, there would have to be a linearly falling pressure along the open surface. But that is impossible because the open surface requires constant pressure.

19.8 First calculate the "tensor product" of the cylindrical gradient operator (C-6) with the velocity field,

$$\boldsymbol{\nabla}\boldsymbol{v} = (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z) v_z(r) \boldsymbol{e}_z = \boldsymbol{e}_r \boldsymbol{e}_z \frac{dv_z}{dr} \ .$$

From this result we immediately recover that the divergence vanishes, $\boldsymbol{\nabla} \cdot \boldsymbol{v} = \mathrm{Tr}[\boldsymbol{\nabla}\boldsymbol{v}] = 0$, as well as the convective acceleration $\boldsymbol{v} \cdot (\boldsymbol{\nabla}\boldsymbol{v}) = \boldsymbol{0}$. Dotting from the left with the gradient we obtain the Laplacian

$$\begin{aligned}\boldsymbol{\nabla}^2\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{v} &= (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot\boldsymbol{e}_r\boldsymbol{e}_z\frac{dv_z}{dr} \\ &= \boldsymbol{e}_z\frac{d^2v_z}{dr^2} + \boldsymbol{e}_z\frac{1}{r}\frac{dv_z}{dr} = \boldsymbol{e}_z\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\end{aligned}$$

19.9 The effective pressure gradient is $G = \rho_0 g_0$ and the Reynolds number (19-24) becomes

$$\mathsf{Re} = \frac{g_0 a^3}{4\nu^2} \tag{19-A1}$$

Solving for a we find

$$a = \left(\frac{4\nu^2}{g_0}\mathsf{Re}\right)^{\frac{1}{3}} \approx 0.07\ \mathsf{mm}\times\mathsf{Re}^{\frac{1}{3}} \tag{19-A2}$$

19.10 The simplest way is to recognize that the mass dimension (kg) contained in ρ_0 and η can only be removed by forming the ratio $\nu = \eta/\rho_0$ of dimension m^2/s. Since Q has dimension of m^3/s, the time unit can only be removed by forming the ratio Q/ν which has dimension of m. Finally, dividing with a, we get the dimensionless number $Q/\nu a$ which is proportional to the Reynolds number.

19.13

$$\mathcal{D} = \pi a^2\Delta p = 8\pi\eta U L f(\mathsf{Re}) \ , \tag{19-A3}$$

$$P = \pi a^2\Delta p U = 8\pi\eta U^2 L f(\mathsf{Re}) \ . \tag{19-A4}$$

19.15 a) Use the no-slip boundary conditions on (19-20). b) The shear stress is

$$\sigma_{zr}(r) = \eta\frac{dv_z}{dr} = -\frac{1}{2}Gr + \frac{\eta A}{r} \ .$$

The total drag per unit of length on the two inner surfaces becomes $\sigma_{zr}(a)2\pi a - \sigma_{zr}(b)2\pi b = \pi G(b^2 - a^2)$.

19.17 a) The pressure at the entrance to the pipe is $p = p_0 + \rho_0 g_0 h$ where p_0 is the air pressure. The effective pressure gradient in the tube is $G = \rho_0 g_0(1 + h/L)$ and using the Hagen-Poiseuille law (19-22) we get

$$Q = -\pi b^2\frac{dh}{dt} = \frac{\pi a^4}{8\eta}\rho_0 g_0\left(1 + \frac{h}{L}\right)$$

b) Solving this equation one gets

$$L + h(t) = (L + h_0)e^{-t/\tau} \ , \qquad \tau = \frac{8b^2 L\nu}{a^4 g_0}$$

Emptying time is $t_0 = \tau\log(1 + h_0/L)$. c) For $h_0 \ll L$ we have $t_0 = \tau h_0/L = 8b^2\nu/a^4 g_0$. The reason is that there is no extra hydrostatic pressure from the water in the tank, but only the gradient due to gravity in the pipe.

