7.8 Write $f(r)(3\cos^2\theta - 1) = \dfrac{f(r)}{r^2}(2z^2 - x^2 - y^2)$ and use $\boldsymbol{\nabla}^2(uv) = u\boldsymbol{\nabla}^2 v + v\boldsymbol{\nabla}^2 u + 2\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v$. Result $g = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr} - 6\dfrac{f}{r^2}$.

8 Surface tension

8.1 The pressure jump across the bubble surface is $\Delta p = 4\alpha/a \approx 20$ Pa. The capillary length is as for massive spheres defined by the length scale where the hydrostatic pressure change inside the bubble matches the pressure jump, $R_c = \sqrt{2\alpha/\rho_0 g_0}$, where ρ_0 is the density of air. Numerically it becomes $R_c = 16$ cm. The bubble radius $a = 3$ cm is much smaller than this, and the bubble should be quite spherical.

8.2 **(a)** Put $x = r\cos\phi$ and $y = r\sin\phi$. A circle with radius R and center at $z = R$ has in the rz-plane the equation $R^2 = r^2 + (z-R)^2 \approx r^2 + R^2 - 2zR$ for $r \ll R$, or $z = r^2/2R$. Comparing with the polynomial one finds $1/R = \partial^2 z/\partial r^2 = 2(a\cos^2\phi + b\sin^2\phi + 2c\cos\phi\sin\phi)$. **(b)** The extrema are determined from the vanishing of $\partial(1/R)/\partial\phi = 2(-(a-b)\sin 2\phi + 2c\cos 2\phi)$, or $\tan 2\phi = (a-b)/2c$. The solutions are $\phi = \phi_0$ and $\phi = \phi_0 + \pi/2$ where $\phi_0 = \frac{1}{2}\arctan[(a-b)/2c]$.

8.3 Expanding to second order around $(x, y, z) = (x_0, 0, z_0)$ we find

$$\Delta z = \alpha\Delta x + \frac{1}{2}\beta\Delta x^2 + \frac{\alpha}{2x_0}y^2 \ , \tag{8-A1}$$

where $\Delta z = z - z_0$, $\Delta x = x - x_0$, $\alpha = f'(x_0) = \tan\theta$, and $\beta = f''(x_0)$. Introduce a local coordinate system with coordinates ξ and η in $(x_0, 0, z_0)$

$$\Delta x = \xi\cos\theta + \eta\sin\theta \tag{8-A2}$$

$$\Delta z = -\xi\sin\theta + \eta\cos\theta \tag{8-A3}$$

Substituting and solving for η keeping up to second order terms,

$$\eta = \frac{1}{2}\beta\cos^3\theta\,\xi^2 + \frac{\sin\theta}{2x_0}y^2 \tag{8-A4}$$

Hence

$$\frac{1}{R_1} = \frac{\partial^2\eta}{\partial\xi^2} = \beta\cos^3\theta \ , \qquad \frac{1}{R_2} = \frac{\partial^2\eta}{\partial y^2} = \frac{\sin\theta}{x_0} \tag{8-A5}$$

But

$$\beta = \frac{d^2 z}{dx^2} = \frac{d\tan\theta}{dx} = \frac{1}{\cos^2\theta}\frac{d\theta}{dx} = \frac{1}{\cos^2\theta}\frac{ds}{dx}\frac{d\theta}{ds} = \frac{1}{\cos^3\theta}\frac{d\theta}{ds} \tag{8-A6}$$

proving that $1/R_1 = d\theta/ds$.

9 Stress

9.1 The normal reaction is the weight N and the tangential reaction is $T = \mu N$. The angle is given by $\tan\alpha = T/N = \mu$.

9.2 The kinetic energy of the car is $\mathcal{T} = \frac{1}{2}mv^2$ and the maximal friction without skidding is $\mathcal{F} = \mu_0 m g_0$. Since the force is constant the braking distance is $d_0 = v^2/2\mu_0 g_0 \approx 44\ m$. Skidding we have $\mathcal{F} = \mu m g_0$, so the distance becomes $d = d_0\mu_0/\mu \approx 56\ m$.

9.3 **(a)** $\sigma = F/NA = 391$ Pa. **(b)** $\sigma = 80{,}000$ Pa $= 0.8$ bar.

9.4 The pressure at the bottom in the middle of the mountain where it is highest is $p \approx \rho g_0 h$ where h is its height. Consequently, the maximal value of h is $\sigma/\rho g_0 = 10$ km. On Mars the maximal height is 27 km.

9.5 The characteristic equation is $-\lambda^3 + 3\tau\lambda^2 = 0$. Eigenvalues $\lambda = 3\tau$ and $\lambda = 0$ (doubly degenerate). Eigenvectors $\boldsymbol{e}_1 = (1,1,1)/\sqrt{3}$, $\boldsymbol{e}_2 = (-2,1,1)/\sqrt{6}$ and $\boldsymbol{e}_3 = (0,-1,1)/\sqrt{2}$, or any linear combination of the last two.

9.6 Let the stress tensor be diagonal in a given coordinate system. Under a small rotation through an angle ϕ

$$x' = x - \phi y \qquad\qquad y' = y + \phi x \tag{9-A1}$$

we find

$$\sigma'_{yx} = \phi\sigma_{xx} - \phi\sigma_{yy} = \phi(\sigma_{xx} - \sigma_{yy}) \tag{9-A2}$$

Since that has to vanish, we must have $\sigma_{xx} = \sigma_{yy}$ and similarly for the other components.

9.7

(a) The body starts to move when the elastic force equals the maximal static friction, *i.e.* $ks = \mu_0 m g_0$ or $s = \mu_0 m g_0/k$.

(b) When the body is at the point x at time t, the actual stretch is $s + vt - x$. The equation of motion becomes

$$m\ddot{x} = k(s + vt - x) - \mu m g_0 \ .$$

(c) Define $y = x - vt - s + \mu m g/k = x - vt - (1-r)s$. Then $m\ddot{y} = -\omega^2 y$ which has the solution $y = A\cos\omega t + B\sin\omega t$. The particular solution follows from the initial conditions $x = \dot{x} = 0$ for $t = 0$.

(d) The velocity is

$$\dot{x} = v(1 - \cos\omega t) + (1-r)s\omega\sin\omega t = 2v\sin^2\frac{\omega t}{2} + 2(1-r)s\omega\sin\frac{\omega t}{2}\cos\frac{\omega t}{2} \ ,$$

which vanishes for the first time after start when

$$\tan\frac{\omega t}{2} = -\frac{(1-r)s\omega}{v} \ ,$$

so that $\omega t_0 = 2\pi - 2\alpha$ where

$$\alpha = \arctan\frac{v}{(1-r)s\omega} \ .$$

The other possibility is $\sin(\omega t/2) = 0$ happens later, for $\omega t = 2\pi$.

