Part II

Hydrostatics

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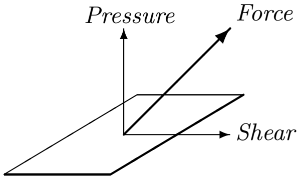
4

Fluids at rest

If the Sun did not shine, if no heat were generated inside the Earth and no energy radiated into space, all the winds in the air and the currents in the sea would die away, and the air and water on the planet would in the end come to rest in equilibrium with gravity. In the absence of external driving forces or time-dependent boundary conditions, and in the presence of dissipative contact forces, any fluid must eventually reach a state of *hydrostatic equilibrium*, where nothing moves anymore anywhere and all fields become constant in time. This state must be the first approximation to the sea, the atmosphere, the interior of a planet or a star.

In a continuous system in mechanical equilibrium there is everywhere a balance between contact forces having zero range and body forces with infinite range. Contact interactions between material bodies or even between parts of the same body take place across *contact surfaces.* A contact force acting on a tiny piece of a surface may in principle take any direction relative to the surface, but can of course be resolved into its normal and tangential components. The normal component is called a *pressure force* and the tangential a *shear force.* Solids and fluids in motion can sustain shear forces, whereas fluids at rest can not.



The force on a small piece of a surface can always be resolved in a normal pressure force and a tangential shear force.

In this chapter we shall establish the basic concepts and formalism for pressure in hydrostatic equilibrium and apply it to the sea and the atmosphere. Along the way we shall recapitulate some basic rules of thermodynamics. In the following three chapters we shall continue the study of hydrostatic equilibrium for fishes, icebergs and ships, the interior of planets and stars, and the shapes of large and small fluid bodies.

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4.1 Pressure

A fluid at rest can as mentioned only sustain pressure forces. If shear forces arise, the fluid will tend to flow towards a new equilibrium without shear. This expresses the most basic property of fluids and may be taken as a definition of what constitutes a fluid at the macroscopic level. In this section we shall take a first look at pressure defined as force per unit of area, discuss its microscopic origins, and analyze a couple of elementary applications.

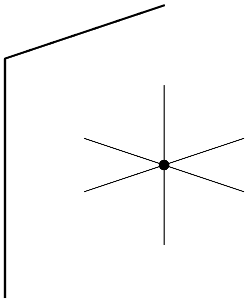
> The SI unit of pressure is pascal ($\mathsf{Pa} = \mathsf{N/m^2} = \mathsf{kg/m/s^2}$), but often pressure is quoted in units of *bars* where 1 $\mathsf{bar} = 10^5$ Pa or in *atmospheres* where 1 atm = 1.01325 bar is close to the average air pressure at sea level. Modern meteorologists are now abandoning these units and tend to quote air pressure in hectopascals (hPa) rather than in millibars.

Microscopic origin of pressure

In a liquid the molecules touch each other and the containing solid walls. The number of molecules that are in direct contact with a small area A of a wall is proportional to the size of this area, and so is the total normal force $\mathcal{F}$ that the molecules collectively exert on the area. The pressure defined as the normal force per unit of area, $p = \mathcal{F}/A$, is thus independent of the size of the small area.

Daniel Bernoulli (1700–82). *Dutch born mathematician who made major contributions to the theory of elasticity, fluid mechanics, and the mechanics of musical instruments.*

A gas consists mostly of vacuum with the molecules moving freely around between collisions, and the pressure on a solid wall arises in this case from the incessant molecular bombardment. We shall estimate the pressure using a simple "molecular" model of a gas going back to Daniel Bernoulli, in which molecules of mass m are only allowed to move back and forth along three orthogonal directions with a fixed velocity v. When a molecule hits a wall orthogonal to one of these directions, it is reflected directly back again and transfers a momentum $2mv$ to the wall. Assuming that all 6 directions of motion are equally probable, the number of molecules hitting an area A of the wall in a small time interval dt will be $dN = \frac{1}{6}\rho A v dt/m$, such that the total momentum transfer to the area in the time dt becomes $d\mathcal{P} = 2mvdN = \frac{1}{3}\rho A v^2 dt$. The normal force exerted on the area equals the rate of momentum transfer, $\mathcal{F} = d\mathcal{P}/dt = \frac{1}{3}\rho A v^2$, and dividing with the area we obtain the pressure,



In Bernoulli's model a molecule moves in any of the 6 possible directions along three orthogonal directions with equal probability .

$$p = \frac{\mathcal{F}}{A} = \frac{1}{3}\rho v^2 . \tag{4-1}$$

In the kinetic theory of gases one finds the same result, except that v is replaced by the root-mean-square average of the molecular velocities, $v = \sqrt{\langle \boldsymbol{v}^2 \rangle}$.

> At normal pressure, $p = 1$ atm, and temperature, $T = 18^\circ$ $\mathsf{C} = 291$ K, the density of air is $\rho \approx 1.2$ $\mathsf{kg/m^3}$, and the molecular velocity $v \approx 500$ $\mathsf{m/s}$. We shall later see that this velocity is related to the velocity of sound, though somewhat larger.

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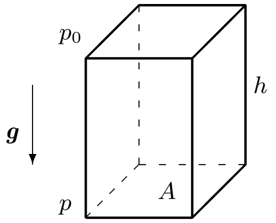
Pressure field

So far we have only defined the pressure acting on the walls of a container. Is it meaningful to speak about pressure in the middle of a fluid away from containing walls? We could of course insert a tiny manometer to measure the pressure, but then we would just obtain the pressure acting on the surface of the manometer, which is another wall. There seems to be no simple way to determine what one would call the true internal pressure in the fluid.

Cutting through these "philosophical" difficulties we shall simply postulate that there is indeed a well-defined pressure field $p(\boldsymbol{x})$ everywhere in the fluid and that it acts along the normal to any surface in the fluid, whether it be a real interface or an imagined cut through the fluid. This postulate is supported by the microscopic view of continuous matter. In a liquid we may define the pressure as the total force per unit of area exerted by the molecules at one side of the cut on the molecules at the other side. Similarly, in a gas the pressure may be defined by the rate of molecular momentum transfer per unit of area across the cut.



A column of sea water. The pressure difference between bottom and top must carry the weight of the water in the box.

Incompressible sea

Consider now a vertical box with cross-sectional area A and height h in a sea with constant density $\rho(\boldsymbol{x}) = \rho_0$. In hydrostatic equilibrium the difference between the pressure forces pA at the bottom and $p_0 A$ at the top must balance the total weight of the water in the box, or

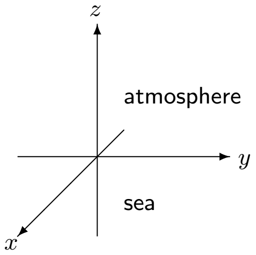
$$pA - p_0 A = \rho_0 A h g_0 \ , \tag{4-2}$$

where g_0 the constant gravity. If this equation were not fulfilled, the total force on the column of water would not vanish, and it would have to move. Dividing by the area A the pressure difference between bottom and top of the box becomes,

$$p - p_0 = \rho_0 g_0 h \ . \tag{4-3}$$

In a *flat-earth coordinate system* with vertical z-axis and the surface of the sea at $z = 0$, we find by setting $h = -z$,

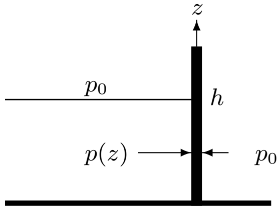
$$\boxed{p = p_0 - \rho_0 g_0 z \ ,} \tag{4-4}$$



The flat-earth coordinate system.

where p_0 is the surface pressure. The linear rise of the pressure with depth $-z$ allows us immediately to calculate the total pressure force on the horizontal and vertical sides of a container. Skew or curved container walls require a more powerful formalism which we shall soon set up.

> Using $\rho_0 \approx 1000$ kg/m^3 and $g_0 \approx 10$ m/s^2, the scale of the pressure increase per unit of depth in the sea becomes $\rho_0 g_0 \approx 10^4$ Pa/m, or about 1 atm/10 m. At the deepest point in the sea, $z \approx -11$ km, the pressure is a little more than 1000 atm. The assumption of constant gravity is well justified even to this depth, because it changes only by about 0.35 %, whereas the density of water changes by a about 4.5 % (see page 69).



Water stemmed up behind a sluice gate. The pressure varies linearly with height z over the bottom.

Example 4.1.1: Water is stemmed up behind a sluice gate of width L to height h. On the water surface and on the outer side of the gate there is atmospheric pressure p_0. On the inside of the gate the pressure is $p(z) = p_0 + \rho_0 g_0 (h - z)$, so that the total force on the gate becomes

$$\mathcal{F} = \int_0^h (p(z) - p_0)\, L dz = \frac{1}{2} \rho_0 g_0 L h^2 \ . \tag{4-5}$$

Because the pressure rises linearly with depth, this result could have been calculated without an explicit integral. The total force is simply the product of the area of the sluice gate Lh with the average pressure excess $\langle p - p_0 \rangle = \frac{1}{2} \rho_0 g_0 h$ acting on the gate.

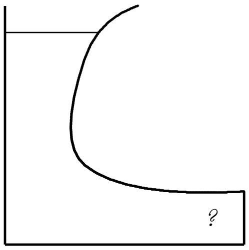
Incompressible atmosphere?

Since air is compressible, it makes little sense to use the above expression (4-4) for the pressure in a fluid with constant density (except for very small values of z). If we anyway do so, we find a pressure which falls linearly with height and reaches zero at a height,

$$z = h_0 = \frac{p_0}{\rho_0 g_0} \ . \tag{4-6}$$

Using $p_0 = 1\ \mathsf{atm}$ and air density $\rho_0 = 1.2\ \mathsf{kg/m^3}$ we get $h_0 = 8.6$ km, which is a tiny bit lower than the height of Mount Everest. This is of course meaningless, since climbers have reached the summit of that mountain without oxygen masks. But as we shall see, this height is nevertheless the correct scale for major changes in the atmospheric properties.

Paradox of hydrostatics



Paradox: the pressure along the vertical wall of the "boot" rises linearly because it has to carry the weight of the water above, but what about the pressure in the tip of the "toe"?

The linear rise of water pressure with depth may, as we have seen, be used to calculate the total pressure force on any vertical container wall. For a curved container wall, like that of a vase or a boot, there seems to be no problem, except handling the necessary mathematics. But if the water column does not reach all the way to the surface, as when you fill a boot with water, what is then the pressure at the flat horizontal bottom? Will it be constant along the bottom as in the open sea, or will it vary? And if it is constant, what is it "up against", since there is only a short column of water above?

The quick answer to this paradox is that the pressure is indeed constant along the horizontal bottom. For if the pressure were lower in the "toe" than in the "heel", there would be unbalanced horizontal pressure forces directed towards the toe acting on a horizontal box of water. But that is not allowed in complete mechanical equilibrium. The only possible conclusion is that the material of the toe of the boot must supply the necessary forces to compensate for the missing weight of the water column.

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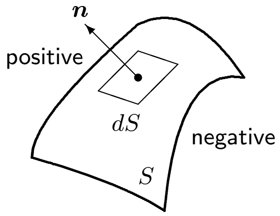
4.2 Formal definition of pressure

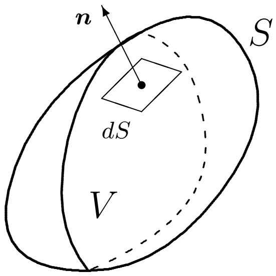
To establish a concise mathematical formalism for pressure we consider a surface S that divides a body into two parts. This surface needs not be a real surface where material properties change dramatically but may just be an imaginary surface separating two parts of the same body from each other. A tiny surface element is characterized by its area dS and the direction of its normal $\boldsymbol{n}$. It is convenient to combine these in the vector surface element

$$d\boldsymbol{S} = (dS_x, dS_y, dS_z) = \boldsymbol{n}\, dS \; . \tag{4-7}$$



All normals to an oriented open surface have a consistent orientation with common positive and negative sides.

There is nothing intrinsic in a surface which defines the orientation of the normal, *i.e.* whether the normal is really $\boldsymbol{n}$ and not $-\boldsymbol{n}$. A choice must, however, be made, and having done that, one may call the side of the surface element into which the normal points, positive (and the other of course negative). Usually neighboring surface elements are required to be oriented consistently, *i.e.* with the same positive sides. By universal convention the normal of a closed surface is chosen to be directed out of the enclosed volume, so that the enclosed volume always lies at the negative side of its surface.

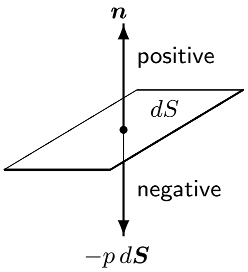


A volume V defined by the closed surface S has all normals oriented towards the outside.

Local and global pressure force

The contact forces due to microscopic molecular interactions have a finite range at the molecular scale, but zero range at macroscopic distances. Across a tiny but still macroscopic piece of surface, the number of neighboring molecules participating in the interaction as well as the force they exert may for this reason be expected to be proportional to the area of the surface. In a fluid at rest the only contact force is the pressure force acting along the normal to the surface, and the force exerted by the material at the positive side of a surface element on the material at the negative side must be of the form

$$\boxed{d\boldsymbol{\mathcal{F}} = -p\, d\boldsymbol{S} \; ,} \tag{4-8}$$



The force on a vector surface element under positive pressure is directed against the normal.

with a coefficient of proportionality p called the *pressure*. Convention dictates that a positive pressure exerts a force directed *towards* the material on the negative side, and that explains the minus sign.

The total pressure force acting on a surface S is obtained by summing up all the little vector contributions from each surface element,

$$\boldsymbol{\mathcal{F}} = -\int_S p\, d\boldsymbol{S} \; . \tag{4-9}$$

This is the force which acts on the cork in the champagne bottle, moves the pistons in the cylinders of your car engine, breaks a dam, and sends off a bullet from a canon. It is also this force that lifts fishes, ships, and balloons and thereby cancels their weight so that they are able to float (see chapter 5).

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Same pressure in all directions?

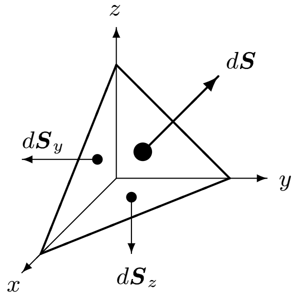
Newton's third law guarantees that the material on the negative side of a surface element reacts with an equal and opposite force, $-d\boldsymbol{\mathcal{F}} = -p\,(-d\boldsymbol{S})$, on the material on the positive side (provided there is no surface tension). Since the surface vector seen from the negative side is $-d\boldsymbol{S}$, the above relation shows that the pressure also has the value p on the negative side of the surface. This is part of a much stronger result, called *Pascal's law*, which we shall prove below: *the pressure in a fluid at rest is independent of the direction of the surface element on which it acts.* It implies that pressure $p(\boldsymbol{x})$ cannot depend on the normal $\boldsymbol{n}$, but only on the location $\boldsymbol{x}$ of a surface element, and is therefore a true scalar field.

Blaise Pascal (1623–1662). *French mathematician and physicist. Founded probability theory. Constructed what may be viewed as the first digital calculator. He spent his later years with religious thinking in the Cistercian abbey of Port-Royal. More than one property of pressure goes under the name of Pascal's law.*

The simple reason for pressure being the same in all directions in hydrostatic equilibrium is that the pressure acts on the surface of a body whereas a body force by definition acts on the volume. If we let the body shrink, the contribution from the body force will vanish faster than the contribution from the surface force because the volume vanishes faster than the surface area. In the limit of vanishing body size only the surface force is left, but it must then itself vanish in hydrostatic equilibrium where the total force on all parts of a body has to vanish. This argument will now be fleshed out in mathematical detail.

∗ **Proof of Pascal's law:** Assume first that the pressure is actually different in different directions. We shall then show that for physical reasons this assumption cannot be maintained. Consider a tiny body in the shape of a tetrahedron with three sides parallel to the coordinate planes. The total pressure force acting on the body is

$$d\boldsymbol{\mathcal{F}} = -p d\boldsymbol{S} - p_x d\boldsymbol{S}_x - p_y d\boldsymbol{S}_y - p_z d\boldsymbol{S}_z \ , \tag{4-10}$$



A body in the shape of a tetrahedron. The vector normals to the sides are all pointing out of the body ($d\boldsymbol{S}_x$ is hidden from view). Any body shape can be built up from sufficiently many and sufficiently small tetrahedrons.

where we have denoted the pressures acting on the different faces of the tetrahedron by p, p_x, p_y, and p_z and the outwards pointing normals by $d\boldsymbol{S}$, $d\boldsymbol{S}_x$, $d\boldsymbol{S}_y$, and $d\boldsymbol{S}_z$. It is sufficient to consider infinitesimal bodies of this kind, because an arbitrary body shape can be put together from these. Each of the three triangles making up the sides of the tetrahedron is in fact the projection of the front face onto that plane. By elementary geometry the areas of the three projected triangles are dS_x, dS_y and dS_z, so that their vector surface elements become $d\boldsymbol{S}_x = (-dS_x, 0, 0)$, $d\boldsymbol{S}_y = (0, -dS_y, 0)$, and $d\boldsymbol{S}_z = (0, 0, -dS_z)$.

Inserting this in the above equation we find the total force

$$d\boldsymbol{\mathcal{F}} = ((p_x - p)dS_x, (p_y - p)dS_y, (p_z - p)dS_z) \ . \tag{4-11}$$

In hydrostatic equilibrium, which is all that we are concerned with here, the contact forces must balance body forces,

$$d\boldsymbol{\mathcal{F}} + \boldsymbol{f}\, dV = \boldsymbol{0} \ , \tag{4-12}$$

where $\boldsymbol{f}$ is the density of body forces.

The idea is now to show that for sufficiently small tetrahedrons the body forces can be neglected and the surface forces $d\boldsymbol{\mathcal{F}}$ must consequently vanish. Consider instead a geometrically congruent tetrahedron with all lengths scaled by a factor λ. Since the volume scales as the third power of λ whereas the surface areas only scale as the second power, the hydrostatic equation for the scaled tetrahedron becomes $\lambda^2 d\boldsymbol{\mathcal{F}} + \lambda^3 \boldsymbol{f}\, dV = \boldsymbol{0}$ or $d\boldsymbol{\mathcal{F}} + \lambda \boldsymbol{f}\, dV = \boldsymbol{0}$. In the limit of $\lambda \to 0$ it follows that the total contact force must vanish, *i.e.* $d\boldsymbol{\mathcal{F}} = 0$, and using (4-11) we find,

$$\boxed{p_x = p_y = p_z = p \ .} \tag{4-13}$$

As promised, the pressure must indeed be the same in all directions.

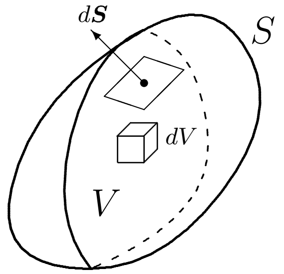
4.3 Hydrostatic equilibrium

In section 4.1 we intuitively used that in a fluid at rest the weight of a vertical column of fluid should equal the difference in pressure forces between bottom and top of the column. We shall now generalize this to an arbitrary macroscopic volume of fluid, often called a *control volume*. The material in a control volume, fluid or solid or whatever, represents the most general "body" that can be constructed in continuum physics.

Up to this point we have studied only two kinds of forces that may act on the material in a control volume. One is a *body force* described by a force density field $\boldsymbol{f}$ caused by long-range interactions, for example gravity $\boldsymbol{f} = \rho \boldsymbol{g}$. The other is a contact force, here the pressure field p, which has zero range and only acts on the surface of the control volume. The total force on the control volume V with surface S is the sum of two contributions

$$\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} \ . \tag{4-14}$$

The first term is for the case of gravity just the weight of the fluid in the volume and the second is the so-called *buoyancy* force. The circle in the symbol for the surface integral is only there to remind us that the surface is closed.



A control volume V with its enclosing surface S, a volume element dV and a surface element $d\boldsymbol{S}$.

Global hydrostatic equilibrium equation

In hydrostatic equilibrium, the total force must vanish for any volume of fluid, $\boldsymbol{\mathcal{F}} = \boldsymbol{0}$, or

$$\boxed{\int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} = \boldsymbol{0}} \ . \tag{4-15}$$

This is the *equation of global hydrostatic equilibrium*, which states that buoyancy must exactly balance the total volume force, *i.e.* the weight. If the cancellation is not exact, as for example when a small volume of water is heated or cooled

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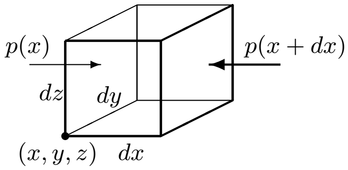
relative to its surroundings, the fluid *must* start to move, either upwards if the buoyancy force is larger than the weight or downwards if it is smaller.

The problem with the global equilibrium equation is that we have to know the fields $\boldsymbol{f}(\boldsymbol{x})$ and $p(\boldsymbol{x})$ in advance to calculate the integrals. Sometimes symmetry considerations can get us a long way. In constant gravity, the sea on the flat Earth ought to have the same properties for all x and y, suggesting that the pressure $p = p(z)$ can only depend on the depth z. This was in fact a tacit assumption used in calculating the pressure in the incompressible sea (4-4), and it is not difficult formally to derive the same result from the equation of global equilibrium (4-15). But in general we need to establish a local form of the equations of hydrostatic equilibrium, valid in each point $\boldsymbol{x}$.

Effective force on material particle



Pressure difference over a small rectangular box.

A material particle is like any other body subject to pressure from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us choose a material particle in the shape of a small rectangular box with sides dx, dy, and dz, and thus a volume $dV = dxdydz$. Since the pressure is slightly different on opposite sides of the box the resultant pressure force is to leading approximation (in the x-direction)

$$d\mathcal{F}_x \approx (p(x,y,z) - p(x+dx,y,z))dydz \approx -\frac{\partial p}{\partial x}\,dxdydz \;.$$

Including the other coordinate directions we obtain

$$d\boldsymbol{\mathcal{F}} = -\boldsymbol{\nabla} p\, dV \;. \tag{4-16}$$

The resultant of all pressure forces acting on a tiny material particle is apparently equivalent to a volume force with a density equal to the negative gradient of the pressure. We shall see below that this result does not depend on the shape of the material particle.

If there is also a true volume force, $\boldsymbol{f}$, for example gravity ($\boldsymbol{f} = \rho\boldsymbol{g}$), acting on the material, the total force on a material particle becomes

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{f}^*\, dV \;,} \tag{4-17}$$

where

$$\boxed{\boldsymbol{f}^* = \boldsymbol{f} - \boldsymbol{\nabla} p \;.} \tag{4-18}$$

This quantity is called the *effective force density.* It must be emphasized that the effective force density is not a true body force, but an expression which for a tiny material particle equals the sum of the true body force and all pressure forces acting on its surface.

Local hydrostatic equilibrium

In hydrostatic equilibrium, the total force on an arbitrary body has to vanish. Applying this to all the material particles in the body, it follows that the effective density of force must vanish everywhere,

$$\boxed{\boldsymbol{f}^* = \boldsymbol{f} - \boldsymbol{\nabla} p = \boldsymbol{0}\ .} \tag{4-19}$$

This is the *local equation of hydrostatic equilibrium.* It is a differential equation valid everywhere in a fluid at rest, and it encapsulates in an elegant way all the physics of hydrostatics.

The flat earth case

Returning to the case of constant gravity in a flat-earth coordinate system we have $\boldsymbol{f} = \rho \boldsymbol{g}_0 = \rho(0, 0, -g_0)$ and the local equilibrium equation takes the following form when written out explicitly in coordinates,

$$\frac{\partial p}{\partial x} = 0\ , \tag{4-20a}$$

$$\frac{\partial p}{\partial y} = 0\ , \tag{4-20b}$$

$$\frac{\partial p}{\partial z} = -\rho g_0\ . \tag{4-20c}$$

The two first equations show that the pressure does not depend on x and y, but only on z, which confirms the previous argument based on symmetry. It also resolves the hydrostatic paradox because we now know that independently of the shape of the container, the pressure will always be the same at a given depth in constant gravity. For the special case of constant density, $\rho(z) = \rho_0$, the last equation may immediately be integrated to yield the previous result (4-4) for the pressure in the incompressible sea.

Constant density

More generally, if the density of the fluid is constant, $\rho = \rho_0$, and the body force is due to gravity, $\boldsymbol{f} = \rho_0 \boldsymbol{g} = -\rho_0 \boldsymbol{\nabla} \Phi$, the equation of hydrostatic equilibrium (4-19) takes the form $-\boldsymbol{\nabla}(\rho_0 \Phi + p) = \boldsymbol{0}$, which implies that

$$H = \Phi + \frac{p}{\rho_0}\ , \tag{4-21}$$

is a constant, independent of $\boldsymbol{x}$. The first term is the gravitational potential and the second term is naturally called the *pressure potential.* There is no agreement in the literature about a name for H, but one might call it the *effective potential*, because the effective density of force $\boldsymbol{f}^* = \rho_0 \boldsymbol{g} - \boldsymbol{\nabla} p = -\rho_0 \boldsymbol{\nabla} H$.

The constancy of H contains the complete solution of the hydrostatic equation. In constant gravity we thus have $\Phi = g_0 z$ and recover immediately the pressure in the sea (4-4) with $H = p_0/\rho_0$.

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Gauss' theorem

Johann Karl Friedrich Gauss (177-1855). *German mathematician of great genius. Contributed to number theory, algebra, non-Euclidean geometry, and complex analysis. In physics he developed the magnetometer. The older (cgs) unit of magnetic strength is named from him.*

The equation of local equilibrium (4-19) has been obtained by applying the global equilibrium equation (4-15) to a tiny material particle. Is it also possible to go the other way and derive the global equation from the local?

The answer to the question is affirmative, because of a purely mathematical theorem due to Gauss (to be proved below), which in its simplest form states that

$$\boxed{\oint_S p\, d\boldsymbol{S} = \int_V \boldsymbol{\nabla} p\, dV\ ,} \tag{4-22}$$

for an arbitrary function $p(\boldsymbol{x})$. Using Gauss theorem it follows immediately that the total force (4-14) equals the integral of the effective density of force,

$$\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} = \int_V \boldsymbol{f}^*\, dV\ . \tag{4-23}$$

One may thus with impunity think of a macroscopic volume of fluid as composed of microscopic material particles, each acted upon by an effective force.

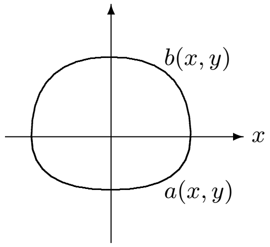
Gauss' theorem is *a fortiori* also valid for a tiny material particle, where the force on a material particle becomes,

$$\oint_S p\, d\boldsymbol{S} = \int_V \boldsymbol{\nabla} p\, dV \approx \boldsymbol{\nabla} p\, V\ .$$

This confirms that the force on a material particle is indeed independent of its shape as long as the pressure gradient is essentially constant across the particle.

Proof of Gauss' theorem: To prove Gauss' theorem, consider first a volume V described by the inequalities $a(x,y) \le z \le b(x,y)$ where $a(x,y)$ and $b(x,y)$ are two functions defined in some area A of the xy-plane. We then find,

$$\begin{aligned}\int_V \nabla_z p\, dV &= \int_A dxdy \int_{a(x,y)}^{b(x,y)} \frac{\partial p(x,y,z)}{\partial z}\, dz \\ &= \int_A dxdy \big(p(x,y,b(x,y)) - p(x,y,a(x,y))\big) \\ &= \oint_S p(x,y,z)\, dS_z\ .\end{aligned}$$



A volume described by $a(x,y) \le z \le b(x,y)$*.*

It is intuitively rather clear that a general volume may be cut up into pieces of this kind. Adding the surface integrals the contributions from the mutual interfaces between neighboring pieces will cancel each other in the sum, leaving only the integral over the outermost surface of the total volume on the left hand side. On the right hand side the volume integrals add up to a volume integral over the total volume of all the pieces. Gauss' theorem thus holds in full generality.

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∗ What about non-gradient forces?

The local equation of hydrostatic equilibrium, $\boldsymbol{f} = \boldsymbol{\nabla} p$, demands that the force density must equal a gradient field, and thus have a vanishing curl $\boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{0}$ (see problem 2.15). A material with constant density, $\rho = \rho_0$, has a gravitational force density $\boldsymbol{f} = \rho_0 \boldsymbol{g} = -\rho_0 \boldsymbol{\nabla}\Phi$ which is evidently a gradient field. But what happens if a force has a manifestly non-vanishing curl? As an example one can take $\boldsymbol{f} = \boldsymbol{a} \times \boldsymbol{x}$, which has $\boldsymbol{\nabla} \times \boldsymbol{f} = 2\boldsymbol{a}$? Then the only possible conclusion is that hydrostatic equilibrium cannot be established, and the fluid must start to move. A physical example of this phenomenon is an electrically charged fluid with a magnetic field that everywhere increases linearly with time. Such a magnetic field induces a static electric field with non-vanishing curl, which indeed accelerates the charged molecules, and thereby the fluid.

4.4 Equation of state

The local equation of hydrostatic equilibrium is not enough in itself, but needs a relation between density and pressure. In the examples of the preceding section we assumed that the fluid was incompressible with constant density and could then integrate the hydrostatic equation and determine the pressure.

Ordinary thermodynamics [67, 6] provides us with a relationship between density ρ, pressure p, and absolute temperature T, called the *equation of state*, which may be written in many equivalent ways, for example

$$f(\rho, p, T) = 0 \; . \tag{4-24}$$

In continuum physics the equation of state should be understood as a *local* relation, valid in every point $\boldsymbol{x}$,

$$f(\rho(\boldsymbol{x}), p(\boldsymbol{x}), T(\boldsymbol{x})) = 0 \; . \tag{4-25}$$

As usual we shall suppress the explicit dependence on $\boldsymbol{x}$ when it does not lead to ambiguity. The actual form of the equation of state for a particular substance is derived from the properties of molecular interactions that fall outside the scope of this book.

Benoit Paul Émile Clapeyron (1799-1864). *French engineer and physicist. Formulated the ideal gas law from previous work by Boyle, Mariotte, Charles, Gay-Lussac, and others. Contributed to early thermodynamics by developing on Carnot's work. Defined the concept of reversible transformations and formulated the first version of the Second Law of Thermodynamics. Established what is now called Clapeyron's formula for the latent heat in the change of state of a pure substance.*

The ideal gas law

The oldest and most famous equation of state is the *ideal gas law*, credited to Clapeyron (1834) and usually presented in the form,

$$pV = nRT \; . \tag{4-26}$$

Here n is the number of moles of gas in a small volume V, and $R = 8.31451(6)$ J/K/mol is the *molar gas constant*. This equation of state has played an enormous role in the development of thermodynamics, and an ideal gas is still the best “laboratory” for understanding materials with a non-trivial thermodynamics. In appendix D the thermodynamics of ideal gases is recapitulated in some detail.

Using that $\rho = M/V = nM_{\mathsf{mol}}/V$, where M is the mass of the gas in V and M_{mol} its molar mass, we obtain the ideal gas law in a form more suited for continuum physics,

$$\boxed{\frac{p}{\rho} = \frac{RT}{M_{\mathsf{mol}}}\ .} \tag{4-27}$$

The ideal gas law is not only valid for pure gases but also for mixtures of pure gases provided one uses the molar average of the molar mass of the mixture (see problem D.1). Real air with $M_{\mathsf{mol}} = 28.9635\ \mathsf{g/cm^3}$ is quite well described by the ideal gas law, although in precise calculations it may be necessary to include non-linear corrections as well as corrections due to humidity [2].

4.5 Barotropic fluid states

The problem with the equation of state (4-24) is, however, that it is not a simple relation between density and pressure which may be plugged into the equation of local hydrostatic equilibrium, but also involves the generally unknown temperature. To solve the general problem of hydrostatic equilibrium we need a further equation connecting temperature, pressure, and density. Such a *heat equation* is also provided by thermodynamics, and we shall derive it in chapter 28.

Barotropic relationship

At this stage it is, however, best to avoid these complications and for simplicity assume that there exists a so-called *barotropic* relationship between pressure and density,

$$p = p(\rho) \qquad \text{or} \qquad \rho = \rho(p)\ . \tag{4-28}$$

The assumption of a barotropic relationship is not as far-fetched as it might seem at first. The condition of constant density $\rho(\boldsymbol{x}) = \rho_0$ which we used in the preceding section to calculate the pressure in the sea is a trivial example of such a relationship.

A less trivial example is obtained if the walls containing a fluid at rest are held at a fixed temperature T_0. The omnipresent heat conduction will eventually cause all of the fluid to attain this temperature everywhere, $T(\boldsymbol{x}) = T_0$, and in this state of *isothermal equilibrium* the equation of state (4-25) simplifies to,

$$f(\rho(\boldsymbol{x}), p(\boldsymbol{x}), T_0) = 0 \tag{4-29}$$

which is indeed a barotropic relationship.

Isothermal atmosphere

Everybody knows that the atmosphere is not at constant temperature, but if we nevertheless assume it to be, we obtain by combining the equation of hydrostatic equilibrium (4-20) with the ideal gas law (4-27)

$$\frac{dp}{dz} = -\rho g_0 = -\frac{M_{\mathsf{mol}}\, g_0}{RT_0} p \ . \tag{4-30}$$

With the initial condition $p = p_0$ for $z = 0$, this ordinary differential equation has the solution

$$p = p_0 e^{-z/h_0} \ , \tag{4-31}$$

where

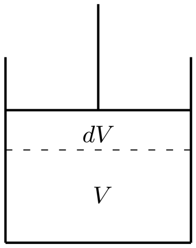
$$h_0 = \frac{RT_0}{M_{\mathsf{mol}}\, g_0} = \frac{p_0}{\rho_0 g_0} \ . \tag{4-32}$$

In the last step we have also used the ideal gas law at $z = 0$ to show that the expression for h_0 is identical to the incompressible scale height (4-6).

In the isothermal atmosphere the pressure thus decreases exponentially with height on a characteristic length scale again set by h_0. Now the pressure at $h = h_0 = 8.6$ km (roughly the top of Mount Everest) is finite and predicted to be $e^{-1} = 37\%$ of the pressure at sea level, or 373 hPa.

Bulk modulus

The archetypal thermodynamics experiment is carried out on a fixed amount $M = \rho V$ of a fluid placed in a cylindrical container with a moveable piston. When you increase the force on the piston a bit, the volume of the chamber decreases, $dV < 0$. The pressure in the fluid must necessarily increase, $dp > 0$. For if this were not the case, an arbitrarily small extra force would send the piston to the bottom of the chamber (and pumping your bicycle would take no effort). Since a larger volume diminishes proportionally more for a given pressure increase, we define the *bulk modulus* as the pressure increase dp per *fractional* decrease in volume $-dV/V$, or



The archetypal thought-experiment in thermodynamics: A cylindrical chamber with a movable piston.

$$K = \frac{dp}{-dV/V} = \frac{dp}{d\rho/\rho} = \rho\frac{dp}{d\rho} \ . \tag{4-33}$$

In the second step we have used the constancy of the mass $dM = \rho dV + V d\rho = 0$ to eliminate the volume. The bulk modulus is a measure of *incompressibility*, and the larger it is, the greater is the pressure increase that is needed to obtain a given fractional increase in density. As a measure of *compressibility* one usually takes $\beta = 1/K$.

The definition of the bulk modulus shows that it is measured in the same units as pressure. In water under normal conditions it is fairly constant, $K \approx$

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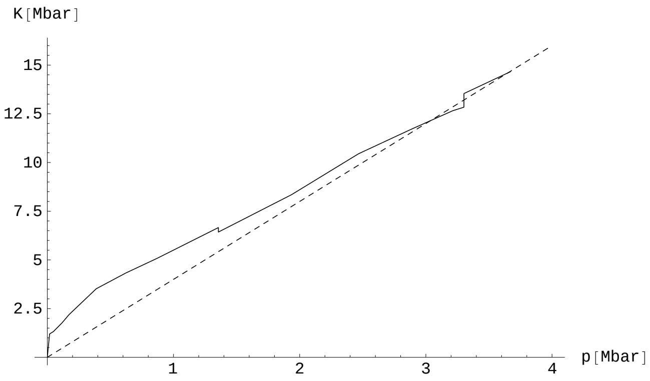

Figure 4.1: *Bulk modulus as a function of pressure in the Earth (data from [2]). The surface of the Earth is to the left and the center to the right in this figure. The bulk modulus varies approximately linearly with pressure, $K \approx 4p$ (the dashed line). The dramatic change in density at the core/mantle boundary (see fig. 3.1) is barely visible in the bulk modulus.*

22,000 atm, doubling only in value between 1 and 3000 atmospheres [2]. As long as the pressure change is much smaller than the bulk modulus, $dp \ll K$, we may estimate the relative change in density from (4-33) to be $d\rho/\rho \approx dp/K$. In the deepest abyss of the sea the pressure is a bit more than 1000 atm, implying that the relative density change is $d\rho/\rho \approx 1/22 \approx 4.5\%$ (see also problem 4.7).

For an isothermal ideal gas it follows from the equation of state (4-27) that the bulk modulus is $K_T = p$ where the index T reminds us that the temperature of the gas must be kept constant. The bulk modulus of the nearly fluid material of the Earth is plotted in fig. 4.1 and varies roughly like $K \approx 4p$.

	K[GPa]
Mercury	25.0
Glycerol	3.94
Water	2.21
Benzene	1.04
Ethanol	0.89
Methanol	0.82
Hexane	0.60

Bulk modulus for common liquids at normal temperature and pressure.

Pressure potential

For a barotropic fluid we may integrate the local equation of hydrostatic equilibrium in much the same way as we did for constant density in (4-21), by defining

$$H = \Phi + w(p) \ , \tag{4-34}$$

where now *pressure potential* is the integral,

$$w(p) = \int \frac{dp}{\rho(p)} \ . \tag{4-35}$$

It follows from the chain rule that $\boldsymbol{\nabla} w(p) = (dw/dp)\boldsymbol{\nabla} p = (1/\rho)\boldsymbol{\nabla} p$ and using local hydrostatic equilibrium (4-19) we obtain $\boldsymbol{\nabla} H = \mathbf{0}$. In hydrostatic equilibrium H is always a constant.

For an ideal gas under isothermal conditions, the pressure potential is calculated by means of the ideal gas law (4-27),

$$w = \int \frac{RT_0}{M_{\mathsf{mol}}\, p} dp = \frac{RT_0}{M_{\mathsf{mol}}} \log p \ . \tag{4-36}$$

In constant gravity $\Phi = g_0 z$, the constancy of H immediately leads back to the pressure in the isothermal atmosphere (4-31).

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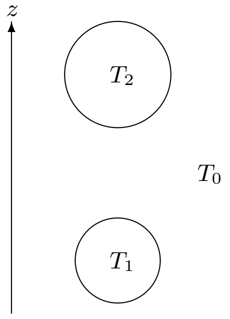
4.6 The homentropic atmosphere

The assumption that the temperature is the same everywhere in the atmosphere is certainly wrong, as anyone who has ever flown in a modern passenger jet can testify. Temperature falls with height instead of staying constant. So the atmosphere is *not* in isothermal equilibrium, and this is perhaps not so surprising, since the "container walls" of the atmosphere, the ground and outer space,have different temperatures. There must be a heat flow through the atmosphere between the ground and outer space, maintained by the inflow of solar radiation and the outflow of geothermal energy. But air is a bad conductor of heat, so although heat conduction does play a role, it is not directly the cause of the temperature drop in the atmosphere.

Of much greater importance are the indirect effects of solar heating, the *convection* which creates air currents, winds, and local turbulence, continually mixing different layers of the atmosphere. The lower part of the atmosphere, the *troposphere*, is quite unruly and vertical mixing happens at time scales that are much shorter than the time scales necessary for reaching thermal equilibrium. There is in fact no true hydrostatic equilibrium state for the real atmosphere. Even if we disregard large-scale winds and weather systems, horizontal and vertical mixing always takes place at small scales, and a realistic model of the atmosphere must take this into account.

Blob-swapping

Let us imagine that we take a small blob of air and exchange it with another blob of air of the same mass, but taken from a different height with a different volume and pressure. In order to fill out the correct volume, one air mass would have to be compressed and the other expanded. If this is done quickly, there will be no time for heat exchange with the surrounding air, and and one air mass will consequently be heated up by compression and the other cooled down by expansion. If the atmosphere initially were in isothermal equilibrium, the temperature of the swapped air would not be the same as the temperature of the surrounding air, and the atmosphere would be brought out of equilibrium.



Swapping air masses from different heights. If the air has temperature T_0 before the swap, the compressed air would be warmer $T_1 > T_0$ and the expanded colder $T_2 < T_0$.

If, however, the surrounding air initially had a temperature distribution, such that the swapped air after the expansion and compression would arrive at precisely the correct temperatures of their new surroundings, a kind of thermodynamic "equilibrium" could be established, in which the omnipresent vertical mixing had essentially no effect. Intuitively, it is reasonable to expect that the end result of fast vertical mixing and slow heat conduction might be precisely such a state. It should however not be forgotten that this state is not a true equilibrium state but rather a dynamically balanced state depending on the incessant small-scale motion in the atmosphere.

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Isentropic processes in ideal gases

A process that takes place without exchange of heat between the system and its environment is said to be *adiabatic*. If furthermore the process is *reversible*, it will conserve the entropy and is called *isentropic*. From the thermodynamics of ideal gases (see appendix D) it follows that an isentropic process in a fixed amount M of an ideal gas will leave the expression pV^γ unchanged. Here γ is the so-called *adiabatic index* which for a gas like air with diatomic molecules is approximately $\gamma \approx 7/5 = 1.4$. In terms of the density $\rho = M/V$ an isentropic process thus obeys,

$$p\,\rho^{-\gamma} = C \; . \tag{4-37}$$

Whereas the "constant" C keeps its value during the isentropic process, it can in principle vary with the position $\boldsymbol{x}$.

Homentropic gas

The atmosphere of the flat Earth is translationally invariant in the horizontal directions, implying that C can only depend on z, and the blob-swapping argument furthermore indicates that C should- also be independent of z. The lower atmosphere, the troposphere, at least approximatively in a so-called *homentropic state* in which (4-37) is valid everywhere, and thus becomes a barotropic relationship with $p = C\rho^\gamma$ everywhere.

The bulk modulus is immediately found to be,

$$K_S = \gamma p \; , \tag{4-38}$$

where the S indicates that the entropy must be kept constant during the compression. The pressure potential (4-35) is similarly obtained,

$$w = \int \frac{dp}{\rho} = \int C\gamma\rho^{\gamma-2}\,d\rho = C\frac{\gamma}{\gamma-1}\rho^{\gamma-1} \; ,$$

and by means of the ideal gas law (4-27) this may also be written,

$$w = \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{RT}{M_{\mathsf{mol}}} \; . \tag{4-39}$$

The surprising result is that the pressure potential is linear in the absolute temperature with a coefficient,

$$c_p = \frac{dw}{dT} = \frac{\gamma}{\gamma-1}\frac{R}{M_{\mathsf{mol}}} \; . \tag{4-40}$$

This constant is the *specific heat at constant pressure* of the gas (see appendix D). For air with $M_{\mathsf{mol}} \approx 29\ \mathsf{g/mol}$ and $\gamma \approx 7/5$ its value is $c_p \approx 1000\ \mathsf{J/K/kg}$.

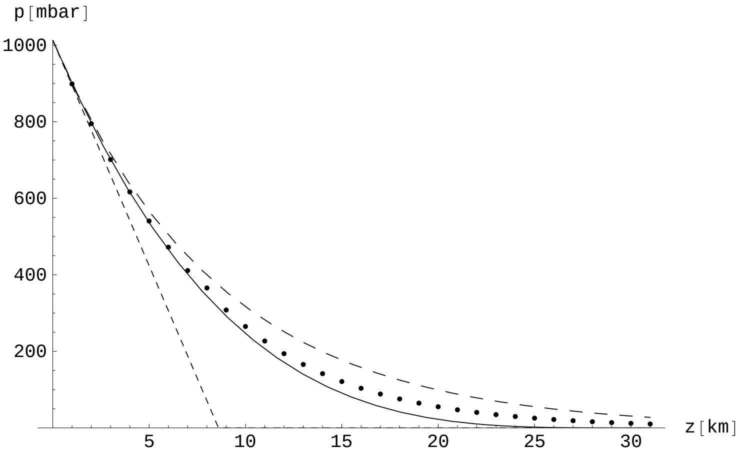

Figure 4.2: *Three different models for the atmospheric pressure: constant density (dashed), homentropic (fully drawn) and isothermal (large dashes), plotted together with the standard atmosphere data (dots)[3]. The parameters are $h_0 = 8.6$ km and $\gamma = 7/5$.*

The atmospheric temperature lapse rate

In constant gravity we find from (4-34) that$H = g_0 z + c_p T$, implying that the temperature drops linearly with height ,

$$T = T_0 - \frac{g_0}{c_p} z \ , \tag{4-41}$$

where T_0 is the temperature at the surface $z = 0$. The magnitude of the vertical temperature gradient $-dT/dz = g_0/c_p \approx 0.01$ K/m = 10 K/km is called the *atmospheric temperature lapse rate.*

Introducing the temperature T_0 at sea level, the above equation may be written

$$T = T_0 \left(1 - \frac{z}{h_2}\right) \ , \tag{4-42}$$

with the scale height,

$$h_2 = \frac{c_p T_0}{g_0} = \frac{\gamma}{\gamma - 1} h_0 \ . \tag{4-43}$$

With $\gamma = 7/5$ we find $h_2 \approx 30$ km. At this altitude the temperature has dropped to absolute zero, which is of course unphysical. It is nevertheless a reasonable scale for the height of the atmosphere.

The equation of state (4-27) combined with the adiabatic law (4-37) implies that $T\rho^{1-\gamma}$ and $p^{1-\gamma}T^{-\gamma}$ are also constant, and the density and pressure become,

$$\rho = \rho_0 \left(1 - \frac{z}{h_2}\right)^{\frac{1}{\gamma-1}} \ , \qquad p = p_0 \left(1 - \frac{z}{h_2}\right)^{\frac{\gamma}{\gamma-1}} \ . \tag{4-44}$$

Both of these quantities vanish like the temperature for $z = h_2$. At the top of Mt. Everest the pressure is predicted to be 437 hPa whereas the typical value is 300 hPa.

The real atmosphere of Earth

In fig. 4.2 the various atmospheric models for the pressure have been plotted together with data for the standard atmosphere [3]. Even if the homentropic model gives the best fit, it fails at higher altitudes. The real atmosphere is in fact much more complicated than any of these models indicates.

Water vapor is always present in the atmosphere and will condense to clouds in rising currents of air. The latent heat of condensation heats up the air, so that the temperature lapse rate becomes smaller than 10 K/km, perhaps more like $6 - 7$ K/km, leading to a somewhat higher estimate for the temperature at the top of Mount Everest. The clouds may eventually precipitate out as rain, and when the dried air afterwards descends again, for example on the lee side of a mountain, the air will heat up at a higher rate than it cooled during its ascent on the windward side and become quite hot, a phenomenon known as *föhn* in the Alps.

The fact that the temperature lapse rate is smaller in the real atmosphere than in the isentropic model has a bearing on the stability of the atmosphere. If a certain amount of air is transported to higher altitude without heat exchange and condensation of water vapor, it will behave like in the isentropic model and become cooler than the surrounding air. Consequently it will also be heavier than the surrounding air and tend to sink back to where it came from. Conversely, if the real temperature lapse becomes larger than in the isentropic model, the atmosphere becomes unstable and strong vertical currents may arise. This can, for example, happen in thunderstorms.

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Problems

4.1 Consider a canal with a dock gate which is 12 m wide and has water depth 9 m on one side and 6 m on the other side.

- **(a)** Calculate the pressures in the water on both sides of the gate at a height z over the bottom of the canal.
- **(b)** Calculate the total force on the gate.
- **(c)** Calculate the total moment of force around the bottom of the gate.
- **(d)** Calculate the height over the bottom at which the total force acts.

4.2 An underwater lamp is covered by a hemispherical glass with a diameter of 30 cm and is placed with its center at a depth of 3 m on the side of the pool. Calculate the total horizontal force from the water on the lamp, when there is air at normal pressure inside.

4.3 Using a manometer, the pressure in an open container filled with liquid is found to be 1.6 atm at a height of 6 m over the bottom, and 2.8 atm at a height of 3 m. Determine the density of the liquid and the height of the liquid surface.

4.4 An open jar contains two non-mixable liquids with densities $\rho_1 > \rho_2$. The heavy layer has thickness h_1 and the light layer on top of it has thickness h_2. a) An open glass tube is lowered vertically into the liquids towards the bottom of the jar. Describe how high the liquids rise in the tube (disregarding capillary effects). b) The open tube is already placed in the container with its opening close to the bottom when the heavy fluid is poured in, followed by the light. How high will the heavy fluid rise in the tube?

4.5 The equation of state due to van der Waals is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \tag{4-45}$$

where a and b are constants. It describes gases and their condensation into liquids. a) Calculate the isothermal bulk modulus. b) Under which conditions can it become negative, and what does it mean?

4.6 The equation of state for water is to a good approximation (for pressures up to 100,000 bar) given by

$$\frac{p + B}{p_0 + B} = \left(\frac{\rho}{\rho_0}\right)^n \tag{4-46}$$

with $B = 3000$ atm, $n = 7$, $p_0 = 1$ atm and $\rho_0 = 1$ g/cm^3. **(a)** Calculate the bulk modulus K for water. **(b)** Calculate the density and pressure in the sea. **(c)** What is the pressure and the relative compression of the water at the deepest point in the sea ($z = -10.924$ km)?

4.7 Calculate the pressure and density in the sea, assuming constant bulk modulus. Show that both quantities are singular at a certain depth and calculate this depth.

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4.8 A vertical plate is inserted into a liquid at rest with constant density ρ_0 in constant gravity g_0. Introduce a coordinate z going vertically down with the pressure defined to vanish for $z = 0$. In the following we denote the vertical area moments by,

$$I_n = \frac{1}{A}\int_A z^n \, dS \ , \qquad\qquad n = 1, 2, \ldots \tag{4-47}$$

where dS is the surface element. The point $z_M = I_1$ is called the area center.

- **(a)** Calculate the pressure in the liquid.
- **(b)** Show that $I_2 \geq I_1^2$.
- **(c)** Calculate the total pressure force on the plate.
- **(d)** Calculate the total moment of force of the pressure forces around $z = 0$.
- **(e)** Show that the point of attack of the pressure forces is found below the area center $z_P \geq z_M$.
- **(f)** A thin isosceles triangle with height h and bottom length b is lowered into the liquid such that its top point is at $z = 0$. Calculate the area center and the point of attack of the pressure forces.

4.9 Determine the form of the pressure across the core/mantle boundary when the bulk modulus is $K \approx \gamma p$ with $\gamma \approx 4$ throughout the Earth (see fig. 4.1).

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