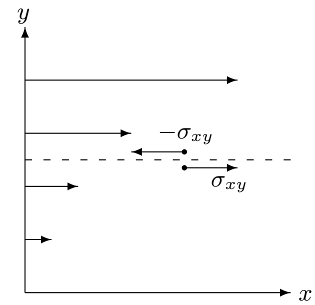
18 Viscosity

All fluids are viscous, except for a component of liquid helium close to absolute zero in temperature. Air, water, and oil all put up resistance to flow, and a part of the money we spend on transport by plane, ship or car goes to overcome fluid friction, and eventually goes to heating the atmosphere, the sea, or the bearings of the car. It may be true that money makes the world go around, but viscosity requires you to have a continuous supply!

It is primarily the interplay between the mechanical inertia of a moving fluid and its viscosity which gives rise to all the interesting and beautiful phenomena, the whirling and the swirling that we are so familiar with. If a volume of fluid inside a larger volume is set into motion, inertia would dictate that it continue in its original motion, were it not checked by the action of internal shear stresses. Viscosity acts as a brake on the free flow of a fluid and will eventually make it come to rest in mechanical equilibrium, unless external driving forces continually supply energy to keep it moving.

In an Aristotelian sense the "natural" state of a fluid is thus at rest with pressure being the only stress component. Disturbing a fluid at rest slightly, setting it into motion with spatially varying velocity field, will to first order of approximation generate stresses that depend linearly on the spatial derivatives of the velocity field. Fluids with a linear relationship between stress and velocity gradients are called Newtonian, and the coefficients in this linear relationship are material constants characterizing viscosity.

In this largely theoretical chapter the formalism for Newtonian viscosity will be set up and we shall finally arrive at the famous Navier-Stokes equations for fluids. Although superficially simple, these non-linear differential equations remain a formidable challenge to both physicists and mathematicians.



If fluid moves faster above a boundary surface, it will exert a positive shear stress on the boundary surface, and conversely.

18.1 Shear viscosity

Consider a fluid flowing steadily along the x-direction with a velocity field $v_x(y)$ which is independent of x, but changes with y. The fluid streams in layers parallel with the xz-plane, and is an example of *laminar* flow. Due to the translational symmetry in the xz-plane there ought to be a constant shear stress $\sigma_{xy}(y)$ acting on any planar surface at a given value of y. A velocity field without y-dependence should not give rise to friction, because the fluid is then in simple uniform motion along the x-axis. If on the other hand the velocity grows with y, *i.e.* for $dv_x(y)/dy > 0$, we expect that the layer immediately above the boundary surface at y will drag along the layer immediately below because of friction and thus exert a positive shear stress, $\sigma_{xy}(y) > 0$, on the boundary surface, and conversely if the velocity decreases with y. It also seems reasonable to expect that a stronger velocity gradient $dv_x(y)/dy$ will evoke a stronger stress.

Such arguments justify that the shear stress in this example should be proportional to the gradient of the velocity field,

$$\boxed{\sigma_{xy}(y) = \eta \frac{dv_x(y)}{dy}} \ . \tag{18-1}$$

This is *Newton's law of viscosity*. Newton did actually not write down this equation but stated it in words in his monumental work *Principia* from 1687. The constant of proportionality, η, is called the coefficient of *shear viscosity*, the *dynamic viscosity*, or simply *the* viscosity. It is a measure of how strongly the fluid layers are coupled by friction and is a material constant of the same nature as the shear modulus for elastic materials. We shall see below that there is also a bulk coefficient of viscosity corresponding to the elastic bulk modulus, but that is rather unimportant in ordinary applications.

The viscosities of naturally occurring fluids range over many orders of magnitude (see table 18.1). Since dv_x/dy has dimension of inverse time, the unit for viscosity η is Pa s (pascal seconds). Although it is sometimes called Poiseuille, there is no special SI-name for it. In the older cgs-system it used to be called poise.

Molecular origin of viscosity in gases

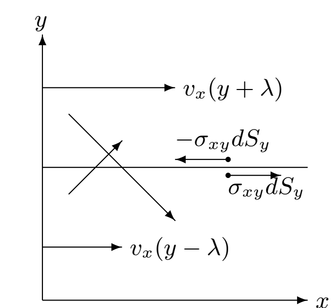
In gases where molecules are far apart, internal stresses are caused by the incessant molecular bombardment of a boundary surface, transferring momentum in both directions across it. In liquids where molecules are in closer contact, internal stress is caused partly by molecular motion as in gases, and partly by intermolecular forces. The resultant stress in a liquid is a quite complicated combination of the two effects, and we shall for this reason limit the following discussion to the molecular origin of stress in gases.

Gas molecules normally move at much greater speeds than the gas itself, and the fluid velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ should as discussed before be understood as the

	density ρ [kg m^{-3}]	dynamic viscosity η [Pa s]	kinematic viscosity ν [m^{2}s^{-1}]
Hydrogen		8.80×10^{-6}	1.10×10^{-4}
Air (NTP)	1.2	1.82×10^{-5}	1.57×10^{-5}
Water	1×10^3	8.90×10^{-4}	8.64×10^{-7}
Ethanol		1.08×10^{-3}	1.08×10^{-3}
Mercury		1.53×10^{-3}	1.14×10^{-7}
Blood		4×10^{-3}	
Engine oil		1.75×10^{-2}	2.03×10^{-5}
Olive oil		6.70×10^{-2}	6.70×10^{-2}
Castor oil		7.00×10^{-1}	7.00×10^{-1}
Glycerol		1.41	1.18×10^{-3}
Ketchup		5×10^1	
Tar		3×10^7	
Glass		1×10^{12}	
Magma			

Table 18.1: *Table of density, and dynamic and kinematic viscosity for common substances (at normal temperature and pressure). Some of the values are only estimates. Notice that air has greater kinematic viscosity than water and hydrogen greater than engine oil!*

center-of-mass velocity of a large collection of molecules. For the case of steady laminar planar flow with increasing velocity field $v_x(y)$, molecules crossing a surface element in the xz-plane from above will carry with them an excess of momentum in the x-direction and therefore exert a force $\mathcal{F}_x$ on the material below. Similarly, the material below will exert an equal and opposite force on the material above.



Layers of fluid moving with different velocities give rise to shear forces because they exchange molecules with different average velocities.

Let the typical distance between molecular collisions in the gas be λ and the typical time between collisions τ. The excess of momentum in the x-direction above an area element dS_y in a layer of thickness λ is of the order of

$$d\mathcal{P}_x \sim (v_x(y+\lambda) - v_x(y))\rho\lambda dS_y \sim \rho\lambda^2 \frac{dv_x(y)}{dy} dS_y \ .$$

This excess of momentum will be carried along by the fast molecular motion in all directions and about half of it will cross the surface in the time τ. The shear stress may be estimated as the momentum transfer per unit of time and area, $\sigma_{xy} = d\mathcal{P}_x/\tau dS_y$, and takes indeed the form of Newton's law of viscosity (18-1) with a rough estimate of the shear viscosity,

$$\eta \sim \rho \frac{\lambda^2}{\tau} \ . \tag{18-2}$$

For gases this estimate becomes of the right order of magnitude (see problem 18.1), but in general it does not yield precise values for the viscosity.

Temperature dependence of viscosity

The viscosity of any material depends on temperature. Common experience from kitchen and industry tells us that most liquids become "thinner" when heated. Gases on the other hand become more viscous at higher temperatures, simply because the molecules move faster at random and thus transport momentum across a surface at a higher rate.

For a gas, the collision length λ may be estimated by requiring that there should be about one other molecule in the cylindrical volume λA swept out between collisions by any molecule of cross section A. Denoting the molecular mass by $m = M_{\mathsf{mol}}/N_A$, this argument leads to the estimate $\rho\lambda A \sim m$, implying that $\rho\lambda$ is independent of both pressure and temperature. But then the viscosity estimate (18-2) can only depend on these quantities through the typical molecular velocity $v_{\mathsf{mol}} \sim \lambda/\tau$. From kinetic gas theory we know that $\frac{1}{2}mv_{\mathsf{mol}}^2 \sim k_B T$ where T is the absolute temperature and k_B is Boltzmann's constant, so that $v_{\mathsf{mol}} \propto \sqrt{T}$, and consequently we must also have $\eta \propto \sqrt{T}$. Thus, if the gas viscosity is η_0 at temperature T_0, it will simply be

$$\eta = \eta_0 \sqrt{\frac{T}{T_0}} \tag{18-3}$$

at temperature T, independently of the pressure.

Kinematic viscosity

The viscosity estimate (18-2) seems to point to another measure of viscosity, called the *kinematic viscosity*[1],

$$\boxed{\nu = \frac{\eta}{\rho}} \,. \tag{18-4}$$

Since its estimate, $\nu \sim \lambda^2/\tau$, does not depend on the unit of mass, this parameter is measured in purely kinematic units of m^2/s (in the older cgs-system, the corresponding unit cm^2/s was called stokes). In fluids with constant density, it is a material constant at equal footing with the dynamic viscosity η (see table 18.1). It should be remembered that in an ideal gas we have $\rho \propto p/T$, so that the kinematic viscosity will depend on both temperature and pressure, $\nu \propto T^{3/2}/p$. For isentropic gases it always decreases with temperature (problem 18.2).

It is as we shall see the kinematic viscosity which appears in the dynamic equations for the velocity field, rather than the dynamic viscosity. Normally, we would think of air as less viscous than water and hydrogen as less viscous than engine oil, but under suitable conditions it is really the other way around. If a flow is driven by inflow of fluid with a certain velocity rather than being controlled by pressure, air behaves as if it is 10–20 times more viscous than water. But subject to a given pressure, air is much easier to set into motion than water because it is a thousand times lighter, and that is what fools our intuition.

[1] The notational clash with Poisson's ratio will in general not be a problem.

18.2 Velocity-driven planar flow

Before turning to the derivation of the complete set of Navier-Stokes equations for viscous flow, we shall explore the concept of shear viscosity a bit further for the simple case of planar flow. Let us as before assume that the flow is laminar and planar with the only non-vanishing velocity component being $v_x = v_x(y,t)$, now also allowing for time dependence. It is rather clear that there can be no advective acceleration in such a field, and formally we also find $(\boldsymbol{v} \cdot \boldsymbol{\nabla})v_x = v_x \nabla_x v_x = 0$. In the absence of volume and pressure forces, the Newtonian shear stress (18-1) will be the only non-vanishing component of the stress tensor, and Cauchy's dynamical equation (15-35) reduces to

$$\rho \frac{\partial v_x}{\partial t} = f_x^* = \nabla_y \sigma_{xy} = \eta \frac{\partial^2 v_x}{\partial y^2} \ ,$$

Dividing by the density (which is assumed to be constant) we get

$$\boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}} \ , \tag{18-5}$$

where ν is the kinematic viscosity (18-4). This is a simplified version of the Navier-Stokes equations, particularly well suited for the discussion of the basic physics of shear viscosity.

Steady planar flow

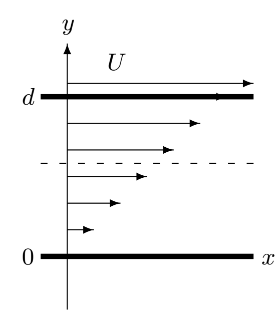
Let us first return to the case of steady planar laminar flow which this chapter began with. In steady flow the left hand side of (18-5) vanishes, and from the vanishing of the right hand side it follows that the general solution must be linear, $v_x = A + By$, with arbitrary integration constants A and B. We shall imagine that the flow is maintained between (in principle infinitely extended) solid plates, one kept at rest at $y = 0$ and one moving with constant velocity U at $y = d$. Where the fluid makes contact with the plates, we require it to assume the same speed as the plates, in other words $v_x(0) = 0$ and $v_x(d) = U$ (this *no-slip* boundary condition will be discussed in more detail later). Solving these conditions we find $A = 0$ and $B = U/d$ such that the field between the plates becomes

$$v_x(y) = \frac{y}{d} U \ , \tag{18-6}$$

independently of the viscosity. From this expression a we obtain the shear stress,

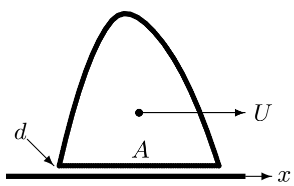
$$\sigma_{xy} = \eta \frac{dv_x}{dy} = \eta \frac{U}{d} \ , \tag{18-7}$$

which is independent of y, as one might have expected.



A Newtonian fluid with spatially uniform properties between moving parallel plates. The velocity field varies linearly between the plates and satisfies the boundary conditions that the fluid is at rest relative to both plates (no-slip).

Viscous friction



A solid object sliding on a plane lubricated surface.

A thin layer of viscous fluid may be used to lubricate the interface between solid objects. From the above solution we may calculate the friction force, or *drag*, exerted on the body by the layer of viscous lubricant (see also chapter 24). Let the would-be contact area between the body and the surface on which it slides be A, and let the thickness of the fluid layer be d everywhere. If the layer is thin, $d \ll \sqrt{A}$, we may disregard edge effects and use the planar stress (18-7) to calculate the drag force,

$$\mathcal{D} \approx -\sigma_{xy} A = -\frac{\eta U A}{d} \ . \tag{18-8}$$

The velocity dependent viscous drag is quite different from the constant drag experienced in solid friction (see section 9.1 on page 142). The decrease in drag with falling velocity makes the object seem to want to slide "forever", and this is presumably what makes ice sports such as skiing, skating, sledging, and curling interesting. A thin layer of liquid water acts here as lubricant. Likewise, it is scary to brake a car on ice, or to aquaplane, because the fall in viscous friction as the speed drops makes the car appear to run away from you.

The quasi-steady horizontal equation of motion for an object of mass M, not subject to other forces than viscous drag, is

$$M \frac{dU}{dt} = -\eta \frac{A}{d} U \ . \tag{18-9}$$

Assuming that the thickness of the lubricant layer stays constant (and that is by no means evident) the solution to (18-9) is

$$U = U_0 e^{-t/t_0} \ , \qquad t_0 = \frac{Md}{\eta A} \ , \tag{18-10}$$

where U_0 is the initial velocity and t_0 is the characteristic exponential decay time for the velocity. Integrating this expression we obtain the total stopping distance

$$L = \int_0^\infty U \, dt = U_0 t_0 = \frac{U_0 M d}{\eta A} \ . \tag{18-11}$$

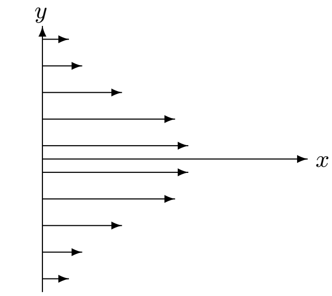
Although it formally takes infinite time for the sliding object to come to a full stop, it does so in a finite distance. The stopping length grows with the mass of the object which is quite unlike solid friction, where the stopping length is independent of the mass. This effect is partially compensated by the dynamic dependence of the layer thickness $d \sim 1/\sqrt{M}$ on the mass (see chapter 24).

Example 18.2.1: In the ice sport of *curling*, a "stone" with mass $M \approx 20$ kg is set into motion with the aim of bringing it to a full stop at the far end of an ice rink of length $L \approx 40$ m. The area of the highly polished contact surface to the ice is $A \approx 700$ cm^2 and the initial velocity about $U_0 \approx 3$ m/s. From (18-11) we obtain the thickness of the fluid layer $d \approx 43$ μm which does not seem unreasonable, and neither does the decay time $t_0 \approx 13$ s. The players' intense sweeping of the ice in front of the moving stone presumably serves to smooth out tiny irregularities in the surface, which could otherwise slow down the stone.

Momentum diffusion

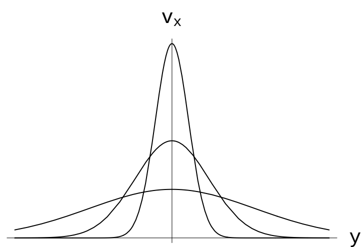
The dynamic equation (18-5) is a typical *diffusion equation* with diffusion constant equal to the kinematic viscosity, ν, also called *momentum diffusivity.* In general, such an equation leads to a spreading of the distribution of the diffused quantity, which in this case is the velocity field, or perhaps better, the momentum density ρv_x. The generic example of a flow with momentum diffusion is the Gaussian "river",

$$v_x(y,t) = U\frac{a}{\sqrt{a^2+4\nu t}}\exp\left(-\frac{y^2}{a^2+4\nu t}\right) , \tag{18-12}$$



Velocity distribution for a planar Gaussian "river" in an "ocean" of fluid.

which may be verified to be a solution to (18-5) by direct insertion, This river starts out at $t = 0$ with Gaussian width a and maximum velocity U, and spreads with time so that it at time t has width $\sqrt{a^2+4\nu t}$. Although momentum diffuses away from the center of the river, the total momentum must remain constant because there are no external forces acting on the fluid. Kinetic energy is on the other hand dissipated and ends up as heat (see problem 18.4).



A Gaussian "river" widens and slows down in the course of time because of viscosity.

For sufficiently large times, $t \gg a^2/4\nu$, the shape of the Gaussian becomes independent of the original width a. This is in fact a general feature of *any* bounded "river" flow: it eventually becomes proportional to $\exp(-y^2/4\nu t)$ (see problem 18.5). The Gaussian factor drops sharply to zero for $y \gtrsim \sqrt{4\nu t}$ and it appears as if momentum diffusion has a fairly well-defined front, which for example may be taken to be $y = 2\sqrt{\nu t}$ where the Gaussian has become $e^{-1} = 37\%$ of its central value. Depending on the application, it is sometimes convenient to choose a more conservative estimate for the spread of momentum, for example $y = 3.5\sqrt{\nu t}$, where the Gaussian factor has dropped to 5% of its central value.

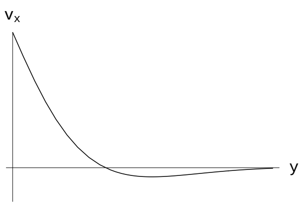
Momentum diffusion may equivalently be characterized by the time, it takes for a velocity disturbance to spread through a distance L by diffusion,

$$t \approx \frac{L^2}{4\nu} , \tag{18-13}$$

or a correspondingly more conservative estimate. It must be emphasized that momentum diffusion (in this case) takes place orthogonally to the general direction of motion of the fluid. In spite of the fact that momentum flows away from the center in the y-direction, there is no mass flow in the y-direction because $v_y = 0$. In less restricted flows there may be a more direct competition between mass flow and diffusion. If the velocity scale of a flow is $|\boldsymbol{v}| \sim U$, it would take the time $t_{\mathsf{flow}} \approx L/U$ for the fluid to move through the distance L, and the ratio of the the diffusion time scale $t_{\mathsf{diff}} \approx L^2/\nu$ to the mass flow time scale becomes a dimensionless number $\mathsf{Re} \approx t_{\mathsf{diff}}/t_{\mathsf{flow}} \approx UL/\nu$, first used by Reynolds to classify different flows. When this number is large compared to unity, momentum diffusion takes much longer time than mass flow and plays only a little role, whereas when Re is small momentum diffusion wins over mass flow and dominates the flow pattern. The Reynolds number is a very useful parameter which will be discussed in more detail in section 18.4.

Shear sound waves

Consider an infinitely extended plate in contact with an infinite sea of fluid. Let the plate oscillate with circular frequency ω, so that its instantaneous velocity in the x-direction is $U(t) = U_0 \cos \omega t$. The motion of the plate is transferred to the neighboring fluid because of the no-slip condition and then spreads into the fluid at large. How far does it go? By direct insertion into (18-5) that



The shape of a transverse wave.

$$v_x(y,t) = U_0 e^{-ky} \cos(ky - \omega t) \ , \qquad k = \sqrt{\frac{\omega}{2\nu}} \ , \tag{18-14}$$

satisfies the planar flow equation (18-5) as well as the no-slip boundary condition $v_x = U(t)$ for $y = 0$. Evidently, this is a damped wave spreading from the oscillating plate into the fluid. Since the velocity oscillations take place in the x-direction whereas the wave propagates in the y-direction it is a *transverse* or *shear* wave. The wave number k both determines the wave length $\lambda = 2\pi/k$ and the decay length of the exponential, also called the *penetration depth* $d = 1/k = \lambda/2\pi$. The wave is critically damped and penetrates less than one wavelength into the fluid, so it is really not much of a wave. Although longitudinal (pressure) waves are also attenuated by viscosity, they propagate over much greater distances (see section 18.6).

Example 18.2.2: A shear sound wave in air of frequency 1000 Hz has wave length 0.4 mm, whereas in water it is 0.1 mm.

18.3 Incompressible Newtonian fluids

There are numerous everyday examples of fluids obeying the Newtonian law of viscosity (18-1), for example water, air, oil, alcohol, and antifreeze. A number of common fluids are only approximatively Newtonian, for example paint and blood, and others are strongly non-Newtonian, for example tomato ketchup, jelly, and putty. There even exist *viscoelastic* materials that are both elastic and viscous, sometimes used in toys that can be deformed like clay but also jump like a rubber ball.

Most everyday fluids are incompressible, or at least effectively so when the flow velocities are much smaller than the velocity of sound (see section 16.4 on page 269). We shall in this section only establish the general dynamical equations for the simpler case of incompressible, isotropic Newtonian fluids and postpone the analysis of the slightly more complicated compressible fluids to section 18.5.

Isotropic viscous stress

Newton's law of viscosity (18-1) is a linear relation between the stress and the velocity gradient, only valid in a particular geometry. As for Hooke's law in elasticity (page 174) we want a more general definition of viscous stress which takes the same form for any geometry and in any Cartesian coordinate system, so that we are free to choose our own reference frame.

Most ordinary fluids are not only Newtonian, but also *isotropic*. Liquid crystals are anisotropic, but so special that we shall not consider them here. In an isotropic fluid at rest there are no directions defined at all and the stress tensor is determined by the pressure, $\sigma_{ij} = -p\,\delta_{ij}$. When such a fluid is set in motion, the velocity field $v_i(\boldsymbol{x}, t)$ defines a direction in every point of space, but as we have argued before the velocity in a point cannot itself provoke stress in the fluid. It is the variation in velocity from point to point that causes stress. Viscous stress is in other words be determined by the tensor of velocity gradients $\nabla_i v_j$.

In an incompressible fluid, the trace of the velocity gradients vanishes, $\sum_i \nabla_i v_i = \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, so that the most general symmetric tensor one can construct from the velocity gradients is of the form,

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + \eta\,(\nabla_i v_j + \nabla_j v_i)}\ . \tag{18-15}$$

The coefficient of the last term may be identified with the shear viscosity η by inserting the field of a steady planar flow $\boldsymbol{v} = (v_x(y), 0, 0)$, because it then follows that the only shear stress is $\sigma_{xy} = \sigma_{yx} = \eta\nabla_y v_x(y)$ in agreement with (18-1). The trace of this stress tensor is $\sum_i \sigma_{ii} = -3p$, in agreement with the general definition of pressure (9-12).

Since a fluid particle is displaced by $\delta\boldsymbol{u} = \boldsymbol{v}\,\delta t$ in a small time interval δt, fluid motion may be seen as a continuous sequence of infinitesimal deformations with strain tensor, $\delta u_{ij} = \frac{1}{2}(\nabla_i \delta u_j + \nabla_j \delta u_i) = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)\,\delta t$. The symmetrized velocity gradients $v_{ij} \equiv \delta u_{ij}/\delta t = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ may thus be understood as the *rate of deformation* or *rate of strain* of the fluid material.

The Navier-Stokes equations for incompressible fluid

The right hand side of Cauchy's general equation of motion (15-35) equals the effective density of force $f_i^* = f_i + \sum_j \nabla_j \sigma_{ij}$. Inserting the stress tensor (18-15) and using again that $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, we find

$$\sum_j \nabla_j \sigma_{ij} = -\nabla_i p + \eta\,(\sum_j \nabla_i \nabla_j v_j + \sum_j \nabla_j^2 v_i) = -\nabla_i p + \eta\boldsymbol{\nabla}^2 v_i\ .$$

Here we have tacitly assumed that the fluid is homogeneous such that the shear viscosity (like the density ρ) does not depend on $\boldsymbol{x}$. If the temperature or chemical composition of the fluid varies in space, the right hand side must be modified.

Inserting this expression into Cauchy's equation of motion and converting to ordinary vector notation we finally obtain the *Navier-Stokes equation* for incompressible fluid (Navier (1822), Stokes (1845))

$$\boxed{\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{g}}\ , \tag{18-16}$$

where $\nu = \eta/\rho$ is the kinematic viscosity and $\boldsymbol{g} = \boldsymbol{f}/\rho$ is the acceleration field of the volume forces (normally due to gravity). The only difference from the Euler equation (16-1) is the second term on the right hand side. Besides the Navier-Stokes equation, we must also impose the divergence condition for incompressibility,

$$\boxed{\nabla \cdot \boldsymbol{v} = 0} \; . \tag{18-17}$$

Given the acceleration field $\boldsymbol{g}$, we now have four equations for the four fields, $\boldsymbol{v}$ and p. Notice, however, that whereas the three velocity fields obey truly dynamic equations with each field having its own time derivative, this is not the case for the pressure which is only determined indirectly through the divergence condition.

Although relatively simple to look at, the Navier-Stokes equations contain all the complexity of real fluid flow, including that of Niagara Falls! It is therefore clear that one cannot in general expect to find simple solutions. Exact solutions can only be found in strongly restricted geometries and under simplifying assumptions concerning the nature of the flow, as in the planar laminar flow examples in the preceding section.

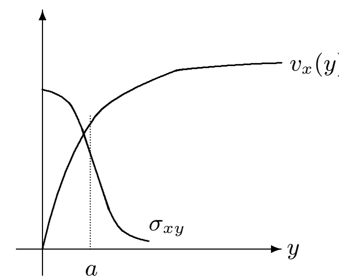
> Among the seven Millenium Prize Problems set out by the Clay Mathematics Institute of Cambridge, Massachusetts, one concerns the existence of smooth, non-singular solutions to the Navier Stokes equations (even for the simpler case of incompressible flow). The prize money of one million dollars illustrates how little we know and how much we would like to know about the general features of these equations which appear to defy the standard analytic methods for solving partial differential equations.

Boundary conditions

Differential equations always require boundary conditions. Field equations that are first order in time, like the Navier-Stokes equation (18-16), need initial values of the fields (and their spatial derivatives) in order to predict their values at later times. But what about physical boundaries, the containers of fluids, or even internal boundaries between different fluids? How do the fields behave there? Let us discuss the various fields that we have met one by one.

Density: The density is easy to dispose of, since it is allowed to be discontinuous and jump at a boundary between two materials, so this provides us with no condition at all. It is evident from the Navier-Stokes equation that a jump in density across a fluid boundary must somehow be accompanied by a jump in the derivatives of the other fields, but we shall not go into this question here.

Pressure: Newton's third law requires the pressure to be continuous across any boundary. This simple picture is, however, complicated by surface tension, which can give rise to a discontinuous jump in pressure across an interface between two materials.



Sketch of rapidly varying velocity and shear stress in a region of size a near a boundary. For $a \to 0$ the velocity develops an abrupt jump and the stress becomes singular. The decrease of the the shear stress away from the discontinuity leads to spreading of a sharp discontinuity.

Velocity: The normal component $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ of the velocity field must be continuous across any boundary, for the simple reason that what goes in on one side must come out on the other. If this were not the case, material would collect at the boundary or holes would develop in the fluid. The latter kind of breakdown can actually happen in extreme situations (cavitation).

The tangential component of velocity $\boldsymbol{v}_t = \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})$ must also be continuous, but for different reasons. The linear relationship (18-1) between stress and velocity gradient implies that a tangential velocity field which changes rapidly along the direction normal to a surface, must create very large and rapidly varying shear stress. In the extreme case of a discontinuous jump in velocity, the shear stress would become infinite. Although large stresses may be created, for example by hitting a fluid container with a hammer, they can however not be maintained for long, but are rapidly smoothed out by viscous momentum diffusion. Only if the continuum approximation breaks down, shear slippage may occur, for example in extremely rarified gases.

Usually the whole velocity field, normal as well as tangential components, will therefore be assumed to be continuous across any boundary between Newtonian fluids. Since a solid wall may be viewed as an extreme Newtonian fluid with infinite viscosity, we recover the previously mentioned *no-slip condition*: a fluid has zero velocity relative to its containing walls. Viscous fluids never slip along the containing boundaries but adhere to them, and this is part of the reason that viscous fluids are *wet*.

∗ Viscous dissipation

If you stir a pot of soup the fluid is set into motion, but eventually it comes to rest again because of internal friction. The work you perform on the soup while stirring it will contribute to its kinetic energy, which in the end — when the motion stops — goes to heat the soup by an immeasurably small amount. We shall discuss heat extensively in chapter 28 but it is useful already at this point to calculate the rate at which kinetic energy is lost to internal friction.

The rate of work of the internal stresses is given by (17-79) on page 317. From the Newtonian stress tensor (18-15) we find the integrand,

$$-\sum_{ij} \sigma_{ij} \nabla_j v_i = -\frac{1}{2} \sum_{ij} \sigma_{ij} v_{ij} = -2\eta \sum_{ij} v_{ij}^2 \tag{18-18}$$

where $v_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ is the strain rate tensor, and where we have used its symmetry to obtain the final expression. Since the final expression is evidently negative, the power of the internal stresses will be negative and always give rise to a loss of kinetic energy, *i.e.* to dissipation.

∗ Non-locality of pressure

For incompressible fluids, the pressure is not given by an equation of state, but rather determined by the divergence condition, and that leads to special difficulties. Calculating the divergence of both sides of (18-16), we obtain a Poisson equation for the pressure,

$$\nabla^2 p = \rho\big(-\nabla\cdot((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) + \nabla\cdot\boldsymbol{g}\big) \ . \tag{18-19}$$

Solutions to the Poisson equation are generally of the same form as the gravitational potential from a mass distribution (3-24) and thus depends non-locally on the field (the source) on the right hand side. Physically this means that any change in the velocity field is instantaneously communicated to the rest of the fluid through the pressure.

Like true rigidity, true incompressibility is an ideal which cannot be reached with real materials, where the velocity of sound sets an upper limit to signal propagation speed. The above result nevertheless means that any local change in the flow will be communicated by the pressure to any other parts of the fluid at the speed of sound. This phenomenon is in fact well-known from everyday experience where the closing of a faucet can result in rather violent so-called "water-hammer" responses from the house piping. The non-locality of pressure is also a major problem in numerical simulations of the Navier-Stokes equations for incompressible fluids where it demands the calculation of a fairly complete solution to the Poisson equation for each numerical step forward in time (see chapter 21).

18.4 Classification of flows

The most interesting phenomena in fluid dynamics arise from the competition between the inertia of the fluid represented in the Navier-Stokes equation (18-16) by the advective term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ and the viscosity represented by $\nu\nabla^2\boldsymbol{v}$. Inertia attempts to continue the motion of a fluid once it is started whereas viscosity acts as a brake. If inertia is dominant we may leave out the viscous term, arriving again at Euler's equation (16-1) describing lively, *inviscid* or *ideal* flow (see chapter 16). If on the other hand viscosity is dominant, we may drop the advective term, and obtain the basic equations for sluggish *creeping* flow (see chapter 20).

The Reynolds number

Osborne Reynolds (1842-1912). *British engineer and physicist. Contributed to fluid mechanics in general, and to the understanding of lubrication, turbulence, and tidal motion, in particular.*

As a measure of how much an actual flow is lively or sluggish, one may make a rough estimate, called the *Reynolds number*, for the magnitude of the ratio of the advective to the viscous terms. To get a simple expression we assume that the velocity is of typical size $|\boldsymbol{v}| \approx U$ and that it changes by a similar amount over a region of size L. The order of magnitude of the first order spatial derivatives of the velocity will then be of magnitude $|\nabla\boldsymbol{v}| \approx U/L$, and the second order derivatives will be $\left|\nabla^2\boldsymbol{v}\right| \approx U/L^2$. Consequently, the Reynolds number becomes

$$\mathsf{Re} \approx \frac{|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|}{|\nu\nabla^2\boldsymbol{v}|} \approx \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} \ . \tag{18-20}$$

	fluid	size $L\,[\mathsf{m}]$	velocity $U\,[\mathsf{ms}^{-1}]$	Reynolds number
submarine	water	100	15	1.7×10^9
airplane	air	50	200	6.3×10^8
blue whale	water	30	10	3.4×10^8
car	air	5	30	9.4×10^6
swimming human	water	2	1	2.3×10^6
running human	air	2	3	3.8×10^5
herring	water	0.3	1	3.8×10^5
golf ball	air	0.043	40	2.2×10^5
ping-pong ball	air	0.040	10	5×10^4
fly	air	0.01	1	600
flea	air	0.001	3	190
gnat	air	0.001	0.1	6
bacterium	water	10^{-6}	10^{-5}	10^{-5}
bacterium	blood	10^{-6}	4×10^{-6}	10^{-9}

Table 18.2: *Table of Reynolds numbers for some moving objects calculated on the basis of typical values of lengths and speeds. Viscosities are taken from table 18.1 on page 329. It is perhaps surprising that a submarine operates at a Reynolds number that is larger than that of a passenger jet at cruising speed, but this is mostly due to the tiny density of air relative to that of water.*

For small values of the Reynolds number, $\mathsf{Re} \ll 1$, advection plays no role and the flow creeps along, whereas for large values, $\mathsf{Re} \gg 1$, viscosity can be ignored and the flow tends to be lively. The streamline pattern of creeping flow is orderly and layered, also called *laminar*, well-known from the kitchen when mixing cocoa into dough to make a chocolate cake (although dough is hardly Newtonian!). The laminar flow pattern continues quite far beyond $\mathsf{Re} \simeq 1$, but depending on the flow geometry and other circumstances, there will be a Reynolds number, typically in the region of thousands, where *turbulence* sets in with its characteristic tumbling and chaotic behavior.

It is often quite easy to estimate the Reynolds number from the geometry and boundary conditions of a flow pattern, as is done in the following examples and in table 18.2.

Example 18.4.1: Getting out of a bathtub you create flows with speeds of say $U \approx 1$ m/s over a distance of $L \approx 1$ m. The Reynolds number becomes $\mathsf{Re} \approx 10^6$ and you are definitely creating visible turbulence in the water. Similarly, when jogging you create air flows with $U \approx 3$ m/s and $L \approx 1$ m, leading to a Reynolds number around 2×10^5, and you know that you must leave all kinds of little invisible turbulent eddies in the air behind you. The fact that the Reynolds number is smaller in air than in water despite the higher velocity is a consequence of the kinematic viscosity being larger for air than for water.

Example 18.4.2: For planar flow between two plates (section 18.1), the velocity scale is set by the velocity difference U between the plates whereas the length scale is set by the distance d between the plates. In the curling example 18.2.1 on page 332 we found $U \approx 3$ m/s and $d \approx 43$ μm, leading to a Reynolds number $\mathsf{Re} = Ud/\nu \approx$ 140. Although not truly creeping flow, it is definitely laminar and not turbulent.

Example 18.4.3: A typical 1/2" water pipe has diameter $d \approx 1.25$ cm and that sets the length scale. If the volume flux of water is $Q = 100$ cm^3/s, the average water speed becomes $U = Q/\pi a^2 \approx 0.8$ m/s and we get a Reynolds number $\mathsf{Re} = Ud/\nu \approx 12{,}000$ which brings the flow well beyond the laminar and into the turbulent regime. For olive oil under otherwise identical conditions we get $\mathsf{Re} \approx 0.15$, and the flow will be creeping.

Hydrodynamic similarity

What does it mean if two flows have the same Reynolds number? A stone of size $L = 1$ m sitting in a steady water flow with velocity $U = 2$ m/s has the same Reynolds number as another stone of size $L = 2$ m in a steady water flow with velocity $U = 1$ m/s. It even has the same Reynolds number as a stone of size $L = 4$ m in a steady airflow with velocity $U = 9$ m/s, because the kinematic viscosity of air is about 18 times larger than of water (at normal temperature and pressure). We shall now see that provided the stones are geometrically similar, *i.e.* have congruent geometrical shapes, flows with the same Reynolds numbers are also *hydrodynamically similar* and only differ by their overall length and velocity scales, so that their flow patterns visualized by streamlines will look identical.

In the absence of volume forces, steady incompressible flow is determined by (18-16) with $\boldsymbol{g} = \boldsymbol{0}$ and $\partial \boldsymbol{v}/\partial t = \boldsymbol{0}$, or

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{v} \ . \tag{18-21}$$

Let us rescale all the variables by means of the overall scales ρ, U, and L, writing

$$\boldsymbol{v} = U\hat{\boldsymbol{v}} \ , \quad \boldsymbol{x} = L\hat{\boldsymbol{x}} \ , \quad p = \rho U^2 \hat{p} \ , \quad \boldsymbol{\nabla} = \frac{1}{L}\hat{\boldsymbol{\nabla}} \ , \tag{18-22}$$

where the hatted symbols are all dimensionless. In terms of these variables, the steady flow equation takes the form,

$$(\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\nabla}})\hat{\boldsymbol{v}} = -\hat{\boldsymbol{\nabla}}\hat{p} + \frac{1}{\mathsf{Re}}\hat{\boldsymbol{\nabla}}^2 \hat{\boldsymbol{v}} \ . \tag{18-23}$$

The only parameter appearing in this equation is the Reynolds number which may be interpreted as the inverse of the dimensionless viscosity. The pressure is as mentioned not an independent dynamic variable and its scale is here fixed by the flow velocity scale, $P = \rho U^2$. If the flow is driven by external pressure differences of magnitude P rather than by velocity, the equivalent flow velocity scale is given by $U = \sqrt{P/\rho}$.

In congruent flow geometries, the no-slip boundary conditions will also be the same, so that any solution of the dimensionless equation can be scaled back to a solution of the original equation by means of (18-22). Thus, the three different flow cases mentioned at the beginning of this subsection may all be obtained from the same dimensionless solution if the stones are geometrically similar and the Reynolds numbers identical.

Even if the flows are similar, the forces exerted on the stones will not be the same. From the estimate of the shear stress $\sigma \approx \eta\,|\boldsymbol{\nabla v}| \approx \eta U/L$ we may estimate the drag on an object of size L to be of magnitude $\mathcal{D} \approx \sigma L^2 = \eta U L = \eta\nu\mathsf{Re}$. Since the Reynolds numbers are the same, the ratio of the drag on the stone in air to the drag on the stone in water is about $\mathcal{D}_{\mathsf{air}}/\mathcal{D}_{\mathsf{water}} \approx (\eta\nu)_{\mathsf{air}}/(\eta\nu)_{\mathsf{water}} \approx 0.37$.

Example 18.4.4 (Flight of the Robofly): The similarity of flows in congruent geometries can be exploited to study the flow around tiny insects by means of enlarged slower moving models, immersed in another fluid. It is, for example, hard to study the air flow around the wing of a hovering fruit fly, when the wing flaps $f = 50$ times per second. For a wing size of $L \approx 4$ mm flapping through 180° the average velocity becomes $U \approx \pi L f \approx 1.3$ m/s and the corresponding Reynolds number $\mathsf{Re} \approx UL/\nu \approx 160$. The same Reynolds number can be obtained (see J. M. Birch and M. H. Dickinson, **Nature** 412, 729 (2001)) from a 19 cm plastic wing of the same shape, flapping once every 6 seconds in mineral oil with kinematic viscosity $\nu = 1.15$ cm^2/s.

Example 18.4.5 (High pressure wind tunnels): In the early days of flight, wind tunnels were extensively used for empirical studies of lift and drag on scaled-down models of wings and aircraft. The smaller geometrical sizes of the models reduced the attainable Reynolds number below that of real aircraft in flight. A solution to the problem was obtained by operating wind tunnels at much higher than atmospheric pressure. Since the dynamic viscosity η is independent of pressure (page 330), the Reynolds number $\mathsf{Re} = \rho UL/\eta$ scales with the air density and thus with pressure. The famous *Variable Density Tunnel (VDT)* built in 1922 by the US National Advisory Committee for Aeronautics (NACA) operated on a pressure of 20 atmospheres and was capable of attaining full-scale Reynolds numbers for models only 1/20'th of the size of real aircraft [54, p. 301]. The results obtained from the VDT had great influence on aircraft design in the following 20 years.

In the presence of external volume forces, for example gravity, or for time dependent inflow, the flow patterns will depend on further dimensionless quantities besides the Reynolds number. We shall only introduce such quantities when they arise naturally in particular cases. Flows in different geometries can only be compared in a coarse sense, even if they have the same Reynolds number. A running man has the same Reynolds number as a swimming herring, and a flying gnat the same Reynolds number as a man swimming in castor oil (which cannot be recommended). In both cases the flow geometries are quite different, leading to different streamline patterns. Here the Reynolds number can only be used to indicate the character of the flow which tends to be turbulent around the running man and laminar around the flying gnat.

18.5 Compressible Newtonian fluids

When flow velocities approach the velocity of sound in a fluid, it is no longer possible to maintain the simplifying assumption of effective incompressibility. Whereas submarines and ships never come near such velocities, passenger jets routinely operate at speeds up to 80% of the velocity of sound, and rockets, military aircraft, the Concorde and the Space Shuttle, are all capable of flying at supersonic and even hypersonic speeds. In all these cases high pressures builds up, especially at the leading edges of the moving bodies.

Shear and bulk viscosity

In compressible fluids the divergence of the velocity field is non-vanishing. This opens up the possibility of adding a term proportional to $(\boldsymbol{\nabla}\cdot\boldsymbol{v})\delta_{ij}$ to the isotropic stress tensor (18-15),

$$\sigma_{ij} = -p\,\delta_{ij} + \eta\,(\nabla_i v_j + \nabla_j v_i) + a\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\delta_{ij}\ . \tag{18-24}$$

Demanding as usual that the pressure is the average of the three normal stresses, $p = -\sum_i \sigma_{ii}/3$, the trace of this expression becomes $-3p = -3p + 2\eta\boldsymbol{\nabla}\cdot\boldsymbol{v} + 3a\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, so that we must have $a = -\frac{2}{3}\eta$. The complete stress tensor thus becomes

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + 2\eta v_{ij}\ ,} \tag{18-25}$$

where v_{ij} is the symmetric velocity gradient tensor,

$$v_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}\,\delta_{ij}\right)\ . \tag{18-26}$$

Since it is traceless, it represents the *shear strain rate*.

The form of the stress tensor (18-25) may be viewed as a first order expansion in the velocity gradients. In the same approximation the pressure may also depend linearly on the velocity gradients, but since the pressure is a scalar it can only depend on the scalar divergence $\boldsymbol{\nabla}\cdot\boldsymbol{v}$, so that the most general form of the pressure must be

$$\boxed{p = p_{\mathsf{e}} - \zeta\boldsymbol{\nabla}\cdot\boldsymbol{v}}\ , \tag{18-27}$$

with coefficients p_{e} and ζ that may depend on the density ρ and temperature T. In hydrostatic equilibrium, $\boldsymbol{v} = \boldsymbol{0}$, the pressure p_{e} is assumed to be given by the equilibrium equation of state, $p_{\mathsf{e}} = p_{\mathsf{e}}(\rho, T)$.

The new parameter ζ is variously called the *bulk viscosity*, the *second viscosity*, or the *expansion viscosity*. Its presence implies that a viscous fluid in motion exerts an extra *dynamic pressure* of size $-\zeta\boldsymbol{\nabla}\cdot\boldsymbol{v}$. The dynamic pressure is negative in regions where the fluid expands ($\boldsymbol{\nabla}\cdot\boldsymbol{v} > 0$), positive where it contracts

($\boldsymbol{\nabla} \cdot \boldsymbol{v} < 0$), and vanishes for incompressible fluids. Bulk viscosity is hard to measure, because one must set up physical conditions such that expansion and contraction become important, for example by means of high frequency sound waves. In the following section we shall calculate the viscous attenuation of sound in fluids, which depends on the bulk modulus. The attenuation of sound is quite complicated and yields a rather frequency dependent bulk viscosity, although it is generally of the same magnitude as the coefficient of shear viscosity.

The Navier-Stokes equations

Inserting the modified stress tensor (18-25) into Cauchy's equation of motion we obtain the field equation,

$$\boxed{\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \eta\left(\boldsymbol{\nabla}^2\boldsymbol{v} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\right) + \boldsymbol{f}} \ . \tag{18-28}$$

This is the most general form of the *Navier-Stokes equation* (Navier (1822), Stokes (1845)). Together with the equation of continuity (15-25), which we repeat here for convenience,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0} \ , \tag{18-29}$$

we have obtained four dynamic equations for the four fields $\boldsymbol{v}$ and ρ, and one constitutive relation (18-27) for the pressure p. As in incompressible fluids, the pressure is not a true dynamic variable.

∗ Viscous dissipation

For compressible fluids the integrand of the power of internal stresses (17-79) is slightly more complicated than for incompressible fluids (18-18). Using the stress tensor (18-25) and the expression for the pressure (18-27), we obtain the integrand of the internal power for compressible fluids,

$$\sum_{ij}\sigma_{ij}\nabla_j v_i = -p_{\mathsf{e}}\boldsymbol{\nabla}\cdot\boldsymbol{v} + \zeta(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + 2\eta\sum_{ij} v_{ij}^2 \ . \tag{18-30}$$

The first term represents the expansion and compression of the fluid against the equilibrium pressure whereas the last two positive definite terms represent the dissipation of kinetic energy through bulk and shear viscosity.

18.6 Viscous attenuation of sound

It has previously (page 334) been shown that free shear waves do not propagate through more than about one wavelength from their origin in any type of fluid. In nearly ideal fluids such as air and water, free pressure waves are as everybody knows capable of propagating over many wavelengths. Viscous dissipation (and many other effects) will nevertheless slowly sap their strength and in the end all of the kinetic energy of the waves will be converted into heat.

In this section we shall calculate the rate of viscous dissipation by finding solutions to the Navier-Stokes equations in the form of damped waves. Alternatively, the rate of dissipation can be calculated from (18-30).

The wave equation

As in the discussion of unattenuated pressure waves in section 16.2 on page 261 we assume to begin with that a barotropic fluid is in hydrostatic equilibrium without gravity, $\boldsymbol{v} = \mathbf{0}$, so that its density $\rho = \rho_0$ and pressure $p = p(\rho_0)$ were constant everywhere. Consider now a disturbance in the form of a small-amplitude motion of the fluid, described by a velocity field $\boldsymbol{v}$ which is so tiny that the non-linear advective term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ can be completely disregarded. This disturbance will be accompanied tiny density corrections, $\Delta\rho = \rho - \rho_0$, and pressure corrections $\Delta p = p - p_0$, which we assume to be of first order in the velocity. Dropping all higher order terms, the linearized Navier-Stokes equations become

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\Delta p + \eta\left(\boldsymbol{\nabla}^2\boldsymbol{v} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\right) , \tag{18-31a}$$

$$\frac{\partial \Delta\rho}{\partial t} = -\rho_0 \boldsymbol{\nabla}\cdot\boldsymbol{v} , \tag{18-31b}$$

$$\Delta p = \frac{K_0}{\rho_0}\Delta\rho - \zeta\boldsymbol{\nabla}\cdot\boldsymbol{v} . \tag{18-31c}$$

The last equation has been obtained from (18-27) by expanding to first order in the small quantities, and using that the equilibrium bulk modulus is $K_0 = \rho\partial p/\partial\rho$ for $\rho = \rho_0$.

Differentiating the second equation after time and making use of the first, we obtain

$$\frac{\partial^2\Delta\rho}{\partial t^2} = \boldsymbol{\nabla}^2\Delta p - \frac{4}{3}\eta\boldsymbol{\nabla}^2\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}^2\Delta p + \frac{4}{3}\frac{\eta}{\rho_0}\boldsymbol{\nabla}^2\frac{\partial\Delta\rho}{\partial t} .$$

Now substituting the pressure correction from the third equation, we arrive at the following equation for the density corrections,

$$\frac{\partial^2\Delta\rho}{\partial t^2} = \frac{K_0}{\rho_0}\boldsymbol{\nabla}^2\Delta\rho + \frac{\zeta + \frac{4}{3}\eta}{\rho_0}\boldsymbol{\nabla}^2\frac{\partial\Delta\rho}{\partial t} . \tag{18-32}$$

If the last term on the right hand side were absent, this would be a standard wave equation of the form (16-6) describing free density (or pressure) waves with phase velocity $c_0 = \sqrt{K_0/\rho_0}$. It is the last term which causes viscous attenuation.

The ratio of the coefficients of the first to the second terms has dimension of inverse time and defines a circular frequency scale,

$$\omega_0 = \frac{K_0}{\zeta + \frac{4}{3}\eta} = \frac{c_0^2\rho_0}{\zeta + \frac{4}{3}\eta} . \tag{18-33}$$

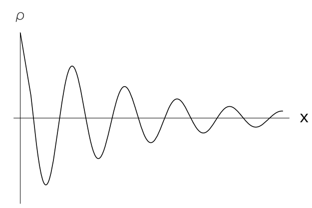
Taking $\zeta \sim \eta$, the right hand side is of the order of $\omega_0 \approx 3 \times 10^9\ \mathsf{s}^{-1}$ in air and $\omega_0 \approx 10^{12}\ \mathsf{s}^{-1}$ in water.In terms of c_0 and ω_0, the wave equation may now be written more conveniently,

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2\Delta\rho}{\partial t^2} = \boldsymbol{\nabla}^2\Delta\rho + \frac{1}{\omega_0}\boldsymbol{\nabla}^2\frac{\partial\Delta\rho}{\partial t}} \ . \tag{18-34}$$

The time derivative in the last term is of order $\omega\Delta\rho$ for a wave with circular frequency ω, and in view of the huge values of the viscous frequency scale ω_0, the ratio of the terms ω/ω_0 will be small, implying that attenuation is weak for normal sound, including ultrasound in the megahertz region.

Damped plane wave

Let us assume that a wave is created by an infinitely extended plane, a "loudspeaker", situated at $x = 0$ and oscillating in the x-direction with a small amplitude at a definite circular frequency ω. The fluid near the plate has to follow the plate and will be alternately compressed and expanded, thereby generating a damped density wave of the form,



Damped density wave.

$$\Delta\rho = \rho_1 e^{-\kappa x}\cos(kx - \omega t) \ , \tag{18-35}$$

where $\rho_1 \ll \rho_0$ is the small density amplitude, k is the wave number, and κ is the *viscous amplitude attenuation coefficient*. In view of the weak attenuation of normal sound, we expect that $\kappa/k \sim \omega/\omega_0 \ll 1$. Inserting this wave into (18-34), we get to lowest order in both κ and ω/ω_0,

$$-\frac{\omega^2}{c_0^2}\cos(kx - \omega t) = -k^2\cos(kx - \omega t) + 2\kappa k\sin(kx - \omega t) - k^2\frac{\omega}{\omega_0}\sin(kx - \omega t) \ .$$

This can only be fulfilled when the wave number has the usual free-wave relation to frequency, $k = \omega/c_0$, and

$$\boxed{\kappa = \frac{k\omega}{2\omega_0} = \frac{\omega^2}{2\omega_0 c_0} = \frac{\omega^2}{2\rho_0 c_0^3}\left(\zeta + \frac{4}{3}\eta\right)} \tag{18-36}$$

The viscous amplitude attenuation coefficient grows quadratically with the frequency, causing high frequency sound to be much more attenuated by viscosity than low frequency sound. In air the viscous attenuation length determined by this expression is huge, about $\kappa^{-1} \approx 50$ km at a frequency of 1000 Hz, but just $\kappa^{-1} \approx 5$ cm at 1 MHz (diagnostic imaging typically uses ultrasound between 1 and 15 MHz). This is also what makes measurements of the attenuation coefficient much easier at high frequencies. From the viscous attenuation coefficient one may in principle extract the value of the bulk viscosity, but this is complicated by several other fundamental mechanisms that also attenuate sound, such as thermal conductivity, and excitation of molecular rotations and vibrations.

In the real atmosphere, many other effects contribute to the attenuation of sound. First of all, sound is mostly emitted from point sources rather than from infinitely extended vibrating planes, and that introduces a drop in amplitude with distance. Other factors like humidity, dust, impurities, and turbulence also contribute, in fact much more than viscosity at the relatively low frequencies that human activities generate (see for example [16, appendix] for a discussion of the basic physics of sound waves in gases).

Problems

18.1 a) Show that in an ideal gas the density of molecules (molecules per unit of volume) is

$$n = \frac{N_A p}{RT} \tag{18-37}$$

and find its value for normal temperature and pressure.

b) Show that in an ideal gas consisting of spherical molecules with diameter d the mean free path between collisions is of the order of

$$\lambda \approx \frac{1}{n\pi d^2} \tag{18-38}$$

Estimate its magnitude in air where the molecular diameter may be taken to be about 0.3 nm (the kinetic theory of gases actually reduces the mean free path by a factor of $\sqrt{2}$).

c) Assume that the molecular velocity $v_0 = \lambda/\tau$ is of the same order of magnitude as the velocity of sound (the kinetic theory of gases actually makes it about a factor $\sqrt{2}$ larger), and use this to estimate the viscosity of air.

18.2 Calculate the temperature dependence of the kinematic viscosity for an isentropic gas. What is the exponent of the temperature it for monatomic, diatomic, and multiatomic gases.

18.3 A car with $M = 1000$ kg moving at $U_0 = 100$ km/h suddenly hits a patch of ice and begins to slide. The total contact area between the wheels and the water is $A = 3200$ cm^2 and it is observed to slide to a full stop in about 300 m. Calculate the thickness of the water layer. Discuss whether it is a reasonable value.

18.4 Consider planar momentum diffusion (page 333) and assume that the flow of the "river" vanishes fast at infinity, as in the Gaussian case. a) Show that for any river flow the total flux of fluid in the z-direction is independent of time. b) Show that the total momentum per unit of length in the z-direction is likewise constant. c) Calculate the kinetic energy per unit of length in the z-direction as a function of time in the Gaussian case.

* **18.5** Show that the general solution to the momentum diffusion equation (18-5) is

$$v_x(y,t) = \frac{1}{2\sqrt{\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-y')^2}{4\nu t}\right) v_x(y',0)\, dy' \tag{18-39}$$

Use this to show that any bounded initial velocity distribution becomes Gaussian for $|y| \to \infty$.

18.6 a) Show that the average of a unit vector $\boldsymbol{n}$ over all directions obeys

$$\langle n_i n_j \rangle = \frac{1}{3}\delta_{ij} \tag{18-40}$$

b) Use this to show that the average pressure 9-12 is also the average of the normal stress acting on an arbitrary surface element in a fluid.

18.7 Estimate the Reynolds number for a) an ocean current, b) a water fall, c) a weather cyclone, d) a hurricane, e) a tornado, f) lava running down a mountainside, and g) plate tectonic motion.

