

# Asymptotic Evaluation of Integrals

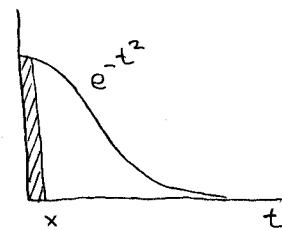
Asymptotic expansion: generate a series which approximates the exact result for small/large parameter

Example:  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dx$

Small  $x$

Use Taylor series:

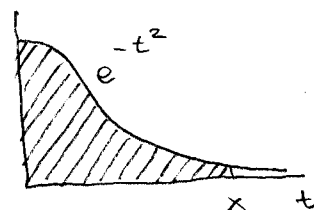
$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \left( 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots \right) dt = \\ &= \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \right), \quad x \ll 1 \end{aligned}$$



Large  $x$

Use integration by parts:

$$\begin{aligned} 1 - \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \\ \int_x^\infty e^{-t^2} dt &= \int_x^\infty \frac{1}{2t} d(e^{-t^2}) = -\frac{e^{-t^2}}{2t} \Big|_x^\infty - \int_x^\infty \frac{e^{-t^2}}{2t^2} dt = \\ &= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \int_x^\infty \frac{3}{4} \frac{e^{-t^2}}{t^4} dt = \dots \end{aligned}$$



$$\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} e^{-x^2} \left( \frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + \dots \right), \quad x \gg 1$$

Note: This series does not converge for any  $x$ , because the general term eventually diverges:

$$(-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n x^{2n-1}} = -4x \left( -\frac{1}{4x^2} \right)^n \frac{(2n-3)!}{(n-2)!} \rightarrow \infty, \quad n \rightarrow \infty$$

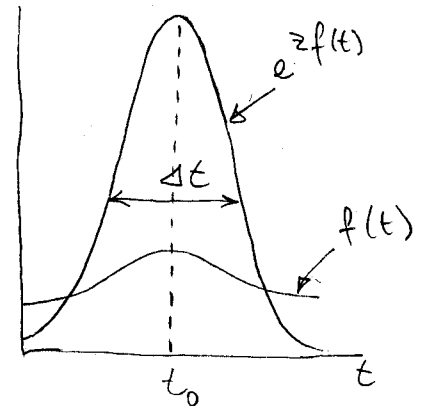
Asymptotic series:  $S_n(x)$  is an asymptotic series expansion of  $f(x)$  ( $S_n(x) \sim f(x)$ ), if  $S_n(x) \rightarrow f(x)$ ,  $x \rightarrow \infty$  for finite  $n$ .

# Saddle-Point (Steepest Descent) Method

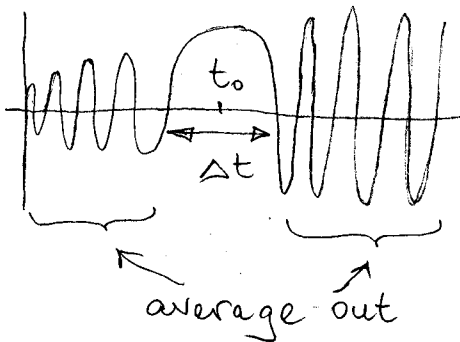
$$I(z) = \int_c g(t) e^{zf(t)} dt, \quad z - \text{real}, \quad z \rightarrow \infty$$

f(t) - real  $\Rightarrow$  Laplace method

Dominant contribution:  $g(t_0) e^{zf(t_0)} \Delta t$



f(t) - imaginary  $\Rightarrow$  Stationary phase method



Dominant contribution:

$$g(t_0) e^{iz\psi_0} \Delta t,$$

$$\psi_0 = \text{Im} f(t_0)$$

What happens when  $f(t)$  is neither real nor imaginary?

Assume  $f(t)$  is analytic:

$$\nabla^2 \text{Re} f = 0 \quad \rightarrow \quad \text{Re} f \text{ has no maxima (or minima)}$$

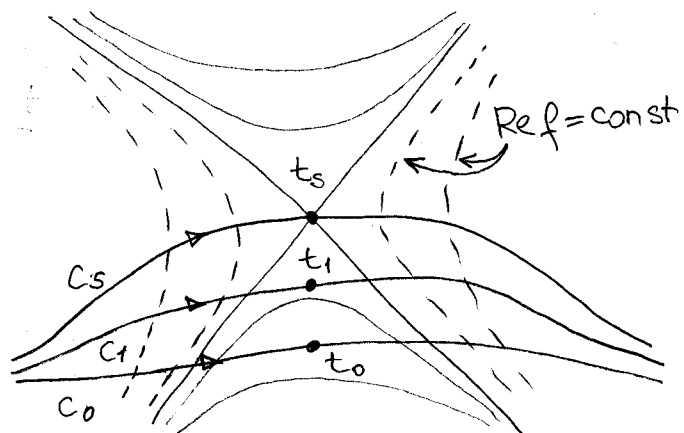
$$\frac{df}{dt} = f' = 0 \quad - \text{ saddle for both } \text{Re} f \text{ and } \text{Im} f.$$

Dominant contribution:

$$I_i \approx g(t_i) e^{z \text{Re} f(t_i)} \int_{c_i} e^{iz \text{Im} f(t)} dt$$

largest for  $c_0$ ,  
smallest for  $c_s$

oscillates fast  
for  $c_0$ , almost  
constant for  $c_s$



In order to remove dependence on oscillations choose path

C with  $\text{Im} f(t) = \text{Im} f(t_s) = \text{const}$  (constant phase)

C ||  $\text{Im} f = \text{const} \Rightarrow$  Cauchy-Riemann condition

C  $\perp$   $\text{Re} f = \text{const} \Rightarrow$

C ||  $\nabla \text{Re} f \Rightarrow$

steepest descent (ascent)

Near  $t_s$ :

$$zf(t) = zf(t_s) + \frac{1}{2}z(t-t_s)^2 f''(t_s) + \frac{1}{3!}z(t-t_s)^3 f'''(t_s) + \dots$$

$$g(t) = g(t_s) + (t-t_s)g'(t_s) + \dots$$

Change integration variable ( $t \rightarrow \tau$ ;  $\Delta\tau = o(1)$ ):

$$(t-t_s) \equiv z^{-1/2} \tau$$

$$\Rightarrow \begin{cases} zf(t) = zf(t_s) + \frac{1}{2}\tau^2 f''(t_s) + o(z^{-1/2}) \\ g(t) = g(t_s) + o(z^{-1/2}) \end{cases}$$

$$I = \int_{\text{saddle } t_s} g(t) e^{zf(t)} dt = \int_{-\infty}^{\infty} e^{zf(t_s)} e^{\frac{1}{2}\tau^2 f''(t_s)} g(t_s) (1 + o(z^{-1/2})) \frac{d\tau}{z^{1/2}} =$$

$$= \left[ \frac{2\pi}{-zf''(t_s)} \right]^{1/2} e^{zf(t_s)} g(t_s) (1 + o(z^{-1/2}))$$

↑  
highest saddle!

### Gamma Function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0 \quad (\text{Laplace transform of } t^{x-1})$$

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1$$

$$\Gamma(x) = \int_0^{\infty} e^{-t} \frac{1}{x} d(t^x) = \frac{1}{x} t^x e^{-t} \Big|_0^{\infty} + \frac{1}{x} \int_0^{\infty} t^x e^{-t} dt = \frac{1}{x} \Gamma(x+1)$$

$$\Rightarrow \Gamma(x+1) = x \Gamma(x)$$

For  $x = n$ -integer:  $\Gamma(n+1) = n \Gamma(n) = \dots = n \cdot (n-1) \cdot \dots \cdot 1 = n!$

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = \int_0^\infty x^{-1} e^{-x^2} 2x dx = 2 \int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

Stirling's Formula

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \int_0^\infty e^{x \ln t - t} dt$$

$$\begin{cases} f(t) = x \ln t - t \\ g(t) = 1 \end{cases} \Rightarrow \begin{cases} f' = \frac{x}{t} - 1 = 0 \Rightarrow t_s = x \\ f'' = -\frac{x}{t^2} = -\frac{1}{x} < 0 \text{ for } x > 0. \end{cases}$$

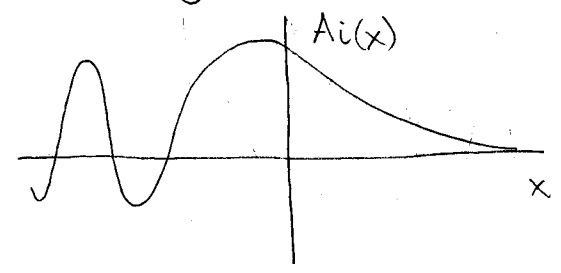
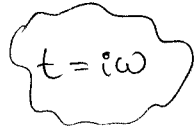
$$\Gamma(x+1) \approx \left( \frac{2\pi}{-(-1/x)} \right)^{1/2} e^{x \ln x - x} = \sqrt{2\pi x} x^x e^{-x}, \quad x \rightarrow \infty$$

$$n! \approx \sqrt{2\pi n} n^n e^{-n} = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \quad n \rightarrow \infty$$

Airy Function

Ai(x) - solution of  $y'' - xy = 0$

$$\begin{aligned} Ai(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega z + \omega^3/3)} d\omega \\ &= \frac{1}{2\pi i} \int_C e^{tz - t^3/3} dt \end{aligned}$$



$$f(t) = tz - \frac{t^3}{3} \Rightarrow f'(t) = z - t^2 = 0 \Rightarrow t_s = \pm \sqrt{z}$$

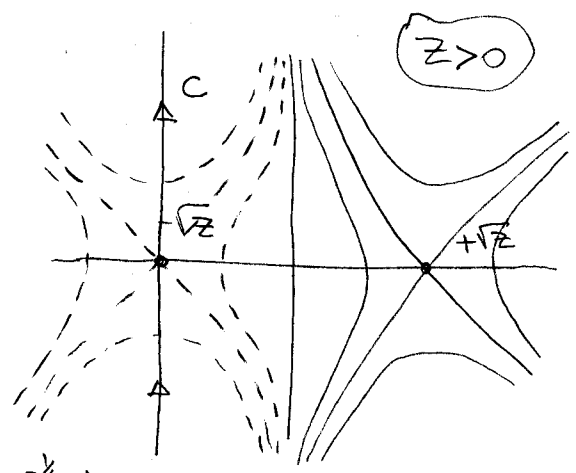
z > 0:  $t_s = -z^{1/2}$

$$f(t_s) = -z^{3/2} + \frac{1}{3} z^{3/2} = -\frac{2}{3} z^{3/2}$$

$$f''(t_s) = -2t = 2z^{1/2}$$

$$f'''(t_s) = -2$$

Change variable:  $t = z^{-1/2} + i z^{-1/4} \tau$



$$Ai(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp\left(-\frac{2}{3} z^{3/2} - \tau^2 - \frac{i}{3} z^{-3/4} \tau^3\right) z^{-1/4} d\tau$$

$$= \frac{\exp(-\frac{2}{3} z^{3/2})}{z^{1/4} 2\pi} \int_{-\infty}^{\infty} e^{-\tau^2} \left( 1 - \frac{i}{3} z^{-3/4} \tau^3 - \frac{1}{18} z^{-3/2} \tau^6 + \dots \right) d\tau =$$

odd

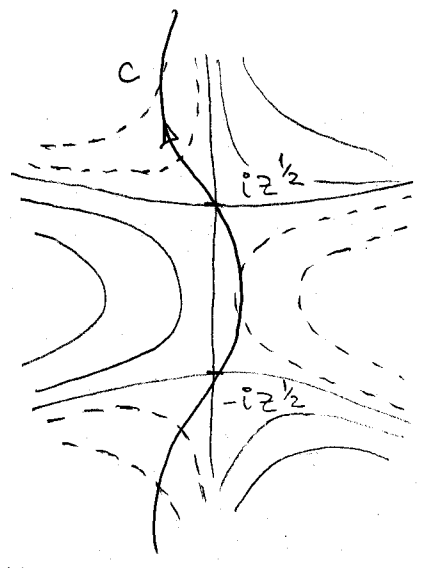
$$Ai(z) = \frac{1}{z^{1/4} 2\pi^{1/2}} e^{-\frac{2}{3}z^{3/2}} \left(1 - \frac{5}{48}z^{-3/2} + \dots\right), \quad z \rightarrow +\infty$$

$z < 0$ :  $t_s = \pm i(-z)^{1/2}$

$$f(t_s) = \pm i \frac{2}{3} (-z)^{3/2} \quad (\text{Re } f(t_s) = 0)$$

$$f''(t_s) = \mp 2i (-z)^{1/2}$$

$$f'''(t_s) = -2$$



Have to go through both saddles.

For each saddle get  $\frac{1}{(\pm i(-z)^{1/2})^{1/2} 2\pi^{1/2}} e^{\pm \frac{2i}{3}(-z)^{3/2}} + \dots$

Combining:  $Ai(z) = \frac{1}{(-z)^{1/4} \pi^{1/2}} \sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right), \quad z \rightarrow -\infty$