Differential Equations

Examples of PDEs in physics:

1) Poisson's equation: $\nabla^2 \varphi = -\rho(\vec{r})$,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- electrostatics: φ - electrostat. potential, ρ - charge density
- gravitation: φ - grav. potential, ρ - mass density

2) Wave equation: $\nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0$

- EM waves in free space, c - speed of light
- acoustic waves in isotropic medium, c - speed of sound

3) Diffusion equation: $k \nabla^2 \varphi = \frac{\partial \varphi}{\partial t} + Q(\vec{r})$

$Q(\vec{r})$ ← heat sources

φ - temperature, $k = \frac{\text{thermal conductivity}}{(\text{specific heat}) \times (\text{density})}$

4) Schrödinger equation: $-\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r}) \Psi = i\hbar \frac{\partial \Psi}{\partial t}$

or, if $\Psi(r,t) = \Psi(r) e^{-iEt/\hbar}$

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r}) \Psi = E \Psi$$

All linear partial differential equations, which have to be supplemented by appropriate boundary / initial conditions.

If bound/init. conditions have certain symmetry, PDEs can often be converted into ODEs.

Separation of variables

All equations above involve Laplacian, ∇. It is separable in 12 (13?) different orthogonal coordinate systems.

Look at 3 most familiar cases

Helmholz equation: $\nabla^2\psi + k^2\psi = 0$

1. Cartesian coordinates

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

Try $\psi(x,y,z) = X(x)Y(y)Z(z)$:

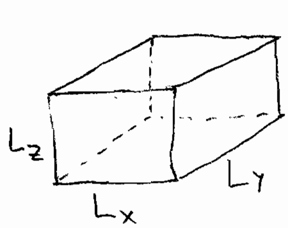
dividing by ψ: $\underbrace{\frac{1}{X}X''(x)}_{\text{function of } x} + \underbrace{\frac{1}{Y}Y''(y)}_{\text{function of } y} + \underbrace{\frac{1}{Z}Z''(z)}_{\text{function of } z} + k^2 = 0$

$$\frac{X''}{X} = c_1, \quad \frac{Y''}{Y} = c_2, \quad \frac{Z''}{Z} = c_3 \quad (c_i - \text{some constants})$$

For consistency: $c_1 + c_2 + c_3 = -k^2$

Cartesian separation makes sense if boundaries have "cartesian" symmetry, e.g. box of size L_x, L_y, L_z.

For instance, if $\psi(x,y,z) = 0$ at $x = 0, L_x$
$y = 0, L_y$
$z = 0, L_z$



$\Rightarrow$ 3 ODEs (for X, Y, Z) of the type

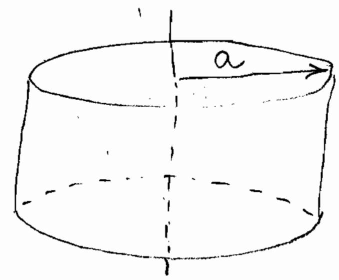
$$X'' = c_1 X, \quad X(0) = X(L_x) = 0$$

General solution: $\boxed{\psi = \sum_{c_1, c_2, c_3} A_{c_1 c_2 c_3} X_{c_1}(x) Y_{c_2}(y) Z_{c_3}(z)}$

2. <u>Cylindrical coordinates</u>: (ρ, φ, z)

$$\nabla^2 \Psi = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho \Psi) + \frac{1}{\rho^2}\partial_\varphi^2 \Psi + \partial_z^2 \Psi$$

Useful for boundary conditions on the surface of a cylinder, e.g., $\Psi(a, \varphi, z) = 0$



Substitute $\Psi = R(\rho)\,\Phi(\varphi)\,Z(z)$ into the Helmholz equation divided by Ψ:

$$\underbrace{\frac{1}{R\rho}\partial_\rho(\rho\,\partial_\rho R) + \frac{1}{\Phi\rho^2}\partial_z^2\Phi}_{\text{function of } \rho, \varphi} + \underbrace{\frac{1}{Z}\partial_z^2 Z}_{\text{function of } z} + k^2 = 0$$

$$\Rightarrow \frac{1}{Z} Z''(z) = C_1$$

Substituting into the equation and multiplying by ρ^2:

$$\underbrace{\frac{\rho}{R}\partial_\rho(\rho\,\partial_\rho R)}_{\text{function of } \rho} + \underbrace{\frac{1}{\Phi}\partial_z^2\Phi}_{\text{function of } \varphi} + \underbrace{\rho^2(C_1 + k^2)}_{\text{function of } \rho} = 0$$

$$\Rightarrow \begin{cases} \frac{1}{\Phi}\Phi''(\varphi) = -C_2 \\ \rho\,\partial_\rho(\rho\,\partial_\rho R) + \rho^2(C_1 + k^2)R - C_2 R = 0 \end{cases} \Leftarrow \text{Bessel's eq.}$$

φ is an angular variable, e.g., $\Phi(\varphi + 2\pi) = \Phi(\varphi)$

$$\Rightarrow \Phi_m(\varphi) = e^{\pm i m \varphi}, \quad \text{so that} \quad C_2 = m^2, \quad m = 0, 1, 2, \ldots$$

"quantization"

General solution: $\boxed{\Psi = \sum_{C_1, m} R_{C_1, m}(r)\,\Phi_m(\varphi)\,Z_{C_1}(z)}$

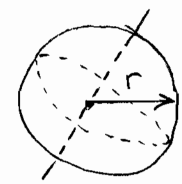
3. Spherical polar coordinates: (r, φ, θ)

φ — azimuthal angle; θ — polar angle

$$\nabla^2\psi = \frac{1}{r^2}\partial_r(r^2\partial_r\psi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\psi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\psi$$

Useful for spherically symmetric boundary conditions, e.g. $\psi(r) = \text{const.}$

Seek a solution in the form $\psi = R(r)\Theta(\theta)\Phi(\varphi)$;



Dividing Helmholz eq. by ψ:

$$\frac{1}{Rr^2}\partial_r(r^2\partial_r R) + \frac{1}{\Theta r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Theta) + \frac{1}{\Phi r^2\sin^2\theta}\partial_\varphi^2\Phi + k^2 = 0.$$

Multiply by $r^2\sin^2\theta$ to "isolate" φ:

$$\underbrace{\frac{\sin^2\theta}{R}\partial_r(r^2\partial_r R) + \frac{\sin\theta}{\Theta}\partial_\theta(\sin\theta\,\partial_\theta\Theta)}_{\text{function of } r,\theta} + \underbrace{\frac{1}{\Phi}\partial_\varphi^2\Phi}_{\text{function of } \varphi} + \underbrace{k^2r^2\sin^2\theta}_{\text{function of } r,\theta} = 0$$

$\Rightarrow \frac{1}{\Phi}\partial_\varphi^2\Phi = -m^2$ (again, φ is an angular variable)

Divide by $\sin^2\theta$ and rearrange:

$$\underbrace{\frac{1}{R}\partial_r(r^2\partial_r R) + r^2k^2}_{\text{function of } r} = \underbrace{-\frac{1}{\Theta\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Theta) + \frac{m^2}{\sin^2\theta}}_{\text{function of } \theta},$$

$$\Rightarrow \begin{cases} \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\Theta) - \frac{m^2}{\sin^2\theta}\Theta + Q\Theta = 0, & Q = \text{const} \\ \partial_r(r^2\partial_r R) + r^2k^2R - QR = 0 & \leftarrow \text{Spherical Bessel eq.} \end{cases}$$

The first eq. above is the associated Legendre equation for $Q = l(l+1)$, $l = 0, 1, 2, \ldots$ (This quantization condition is also a result of θ being an angular variable.)

General solution: $$\boxed{\Psi = \sum_{m,\ell} R_{\ell,m}(r)\,\underbrace{\Theta_\ell(\theta)\,\Phi_m(\varphi)}_{\text{spherical harmonics } Y_{\ell,m}(\theta,\varphi)}}$$

Boundary Value Problem and Differential Operators

After separating variables we always got 2^{nd} order ODEs of the form:

$$P_0(x)u''(x) + P_1(x)u'(x) + P_2(x)u(x) = \rho(x), \quad a \le x \le b$$

with boundary conditions

$$\begin{cases} A_1 u(a) + B_1 u'(a) = C_1 \\ A_2 u(b) + B_2 u'(b) = C_2 \end{cases}$$

← this is the most general form of b.c.'s which preserves linearity.

This construction is called the "boundary value problem",

$$\mathcal{L}[u] = \rho(x),$$

where the linear operator $\mathcal{L}$ is the differential operator plus the boundary conditions.

If $\mathcal{L}$ were a matrix and u – a vector, we would obtain the solution in the form

$$u = \mathcal{L}^{-1}[\rho(x)]$$

The case of diff. operators is not as simple, but the analogy with matrices can be pursued usefully, if we introduce the "matrix element" constuction:

$$\langle v|\mathcal{L}|u\rangle = \int_a^b v(x)\,\mathcal{L}\,u(x)\,dx$$

↑ definition of the scalar product in the space of L_2 functions.