Solution 1 Helmholtz eqn: $\nabla^2\psi + k^2\psi = 0$

In spherical coordinates:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

Let $\psi = R(r)\,\Theta(\theta)\,\phi(\varphi)$

Dividing by ψ

$$\frac{1}{Rr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\, r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\phi r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} + k^2 = 0$$

multiply by r^2

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + k^2r^2}_{\substack{\text{Independent of } (\theta,\phi)\\ = K = \text{constant}}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\phi\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}}_{\substack{\text{Independent of } r\\ = K = \text{constant}}} = 0$$

$$\boxed{\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + r^2k^2R - KR = 0} \rightsquigarrow \text{spherical bessel.}$$

also $$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\phi\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = K$$

multiply by $\sin^2\theta$

$$\underbrace{\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - K\sin^2\theta}_{\text{Independependent of }\varphi} + \underbrace{\frac{1}{\phi}\frac{\partial^2\phi}{\partial\varphi^2}}_{\text{independent of }\theta} = 0$$

$$\therefore \boxed{\frac{1}{\phi}\frac{\partial^2\phi}{\partial\varphi^2} = -m^2}$$

$$\& \boxed{\frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)}{\partial\theta} - \frac{m^2}{\sin^2\theta}\Theta + K\Theta = 0}$$

Problem 2

```
> eq := diff(x(t), t) + x(t) + ε·x(t)^2;
```

$$eq := \frac{d}{dt}\, x(t) + x(t) + \varepsilon\, x(t)^2 \qquad (1)$$

```
> # (a) Look for solution in the form of regular perturbation expansion :
> s0 := x(t) = x0(t) + ε·x1(t) + ε^2·x2(t);
```

$$s0 := x(t) = x0(t) + \varepsilon\, x1(t) + \varepsilon^2\, x2(t) \qquad (2)$$

```
> eqs := series(subs(s0, eq), epsilon, 3);
```

$$eqs := \mathrm{D}(x0)(t) + x0(t) + \left(\mathrm{D}(x1)(t) + x0(t)^2 + x1(t)\right)\varepsilon + \left(\mathrm{D}(x2)(t) + x2(t) + 2\, x0(t)\, x1(t)\right)\varepsilon^2 + \mathrm{O}\left(\varepsilon^3\right) \qquad (3)$$

```
> # At O(1) we find :
> c0 := coeff(eqs, epsilon, 0);
  sl := {dsolve({c0, x0(0) = 1}, x0(t))};
```

$$c0 := \mathrm{D}(x0)(t) + x0(t)$$

$$sl := \{x0(t) = \mathrm{e}^{-t}\} \qquad (4)$$

```
> # At O(ε) we find :
> c1 := subs(sl, coeff(eqs, epsilon, 1));
  sl := sl union {dsolve({c1, x1(0) = 0}, x1(t))};
```

$$c1 := \mathrm{D}(x1)(t) + \left(\mathrm{e}^{-t}\right)^2 + x1(t)$$

$$sl := \{x0(t) = \mathrm{e}^{-t},\, x1(t) = \left(\mathrm{e}^{-t} - 1\right)\mathrm{e}^{-t}\} \qquad (5)$$

```
> # At O(ε^2) we find :
> c2 := subs(sl, coeff(eqs, epsilon, 2));
  sl := sl union {dsolve({c2, x2(0) = 0}, x2(t))};
```

$$c2 := \mathrm{D}(x2)(t) + x2(t) + 2\left(\mathrm{e}^{-t}\right)^2\left(\mathrm{e}^{-t} - 1\right)$$

$$sl := \{x0(t) = \mathrm{e}^{-t},\, x2(t) = \left(\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1\right)\mathrm{e}^{-t},\, x1(t) = \left(\mathrm{e}^{-t} - 1\right)\mathrm{e}^{-t}\} \qquad (6)$$

```
> # So that the perturbative solution is:
> subs(sl, s0);
```

$$x(t) = \mathrm{e}^{-t} + \varepsilon\left(\mathrm{e}^{-t} - 1\right)\mathrm{e}^{-t} + \varepsilon^2\left(\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1\right)\mathrm{e}^{-t} \qquad (7)$$

```
> #` `(b) Exact solution is :
> sle := dsolve({eq, x(0) = 1}, x(t)) : sle;
```

$$x(t) = \frac{1}{-\varepsilon + \mathrm{e}^{t} + \mathrm{e}^{t}\,\varepsilon} \qquad (8)$$

```
> #` `(c) Expanding in small ε:
> x(t) = simplify(series(rhs(sle), epsilon, 3));
```

$$x(t) = \mathrm{e}^{-t} - \left(-1 + \mathrm{e}^{t}\right)\mathrm{e}^{-2t}\,\varepsilon + \mathrm{e}^{-3t}\left(-1 + \mathrm{e}^{t}\right)^2\varepsilon^2 + \mathrm{O}\left(\varepsilon^3\right) \qquad (9)$$

```
> # we find the same result as in (a) - check.
```