

Mathematical Methods of Physics I

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Homework #9

due Tuesday November 1 2011, in class

== show all your work for maximum credit,
 == put labels, title, legends on any graphs
 == acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem 3) More holomorphic mappings, Needham, pp. 211-213

- (a) **(optional)** Use the Cauchy-Riemann conditions to verify that the mapping $z \mapsto \bar{z}$ is not holomorphic.
- (b) The mapping $z \mapsto z^3$ acts on an infinitesimal shape and the image is examined. It is found that the shape has been rotated by π , and its linear dimensions expanded by 12. Determine the possibilities for the original location of the shape?
- (c) Consider the map $z \mapsto \bar{z}^2/z$. Determine the geometric effect of this mapping. By considering the effect of the mapping on two small arrows emanating from a typical point z , one arrow parallel and one perpendicular to z , show that the map fails to produce an *amplitwist*.
- (d) The interior of a simple closed curve \mathcal{C} is mapped by a holomorphic mapping into the exterior of the image of \mathcal{C} . If z travels around the curve counterclockwise, which way does the image of z travel around the image of \mathcal{C} ?
- (e) Consider the mapping produced by the function $f(x + iy) = (x^2 + y^2) + i(y/x)$.
 - (i) Find and sketch the curves that are mapped by f into horizontal and vertical lines. Notice that f appears to be conformal.
 - (ii) Now show that f is *not* in fact a conformal mapping by considering the images of a pair of lines (e.g., one vertical and one horizontal).
 - (iii) By using the Cauchy-Riemann conditions confirm that f is not conformal.
 - (iv) Show that no choice of $v(x, y)$ makes $f(x + iy) = (x^2 + y^2) + iv(x, y)$ holomorphic.
- (f) **(optional)** Show that if f is holomorphic on some connected region then each of the following conditions forces f to reduce to a constant:
 - (i) $\operatorname{Re} f(z) = 0$; (ii) $|f(z)| = \text{const.}$; (iii) $\bar{f}(z)$ is holomorphic too.
- (g) **(optional)** Suppose that the holomorphic mapping $z \mapsto f(z)$ is expressed in terms of the modulus R and argument Φ of f , i.e.,

$$f(z) = R(x, y) \exp i\Phi(x, y).$$

Determine the form of the Cauchy-Riemann conditions in terms of R and Φ .

- (h) (i) By sketching the image of an infinitesimal rectangle under a holomorphic mapping, determine the local magnification factor for the area and compare it with that for an infinitesimal line. Re-derive this result by examining the Jacobian determinant for the transformation.
- (ii) Verify that the mapping $z \mapsto \exp z$ satisfies the Cauchy-Riemann conditions, and compute $(\exp z)'$.
- (iii) Let S be the square region given by $A - B \leq \operatorname{Re} z \leq A + B$ and $-B \leq \operatorname{Im} z \leq B$ with A and B positive. Sketch a typical S for which $B < A$ and sketch the image \tilde{S} of S under the mapping $z \mapsto \exp z$.
- (iv) Deduce the ratio $(\text{area of } \tilde{S})/(\text{area of } S)$, and compute its limit as $B \rightarrow 0^+$.
- (v) Compare this limit with the one you would expect from part (i).

Optional problems

Problem 1) Holomorphic functions, Ahlfors, p. 28

- (a) **(optional)** If $g(w)$ and $f(z)$ are holomorphic, show that $h(z) \equiv g(f(z))$ is, too.
- (b) **(optional)** Verify the Cauchy-Riemann conditions for the holomorphic functions z^2 and z^3 .
- (c) Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$ (with a, b, c and d real). Determine the conjugate harmonic function by integration of the Cauchy-Riemann conditions.
- (d) **(optional)** Show that the functions $\bar{f}(z)$ and $f(\bar{z})$ are simultaneously holomorphic.
- (e) **(optional)** Show that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

Problem 2) Partial fractions, Ahlfors, p. 33

- (a) If Q is a polynomial with distinct roots $\alpha_1, \dots, \alpha_N$ and P is a polynomial of degree less than N , show that the rational function $P(z)/Q(z)$ has the partial fraction development

$$\sum_{n=1}^N \frac{P(\alpha_n)}{(z - \alpha_n) Q'(\alpha_n)}.$$

- (b) Develop in partial fractions the functions $z^4/(z^3 - 1)$ and $1/z(z + 1)^2(z + 2)^3$.
- (c) **(optional)** What is the general form of a rational function that has modulus unity on the circle $|z| = 1$? How are the zeros and poles related to one another? If a rational function is real on $|z| = 1$, how are the zeros and poles situated?

Problem 4) Yet more holomorphic mappings, Needham, pp. 258, 264

- (a) (i) **(optional)** Show that if $f = u + iv$ is a holomorphic mapping then $(\nabla u) \cdot (\nabla v) = 0$, where ∇ is the two-dimensional gradient operator of ordinary vector calculus. Explain the geometrical content of this result.
- (ii) **(optional)** Show that both the real and the imaginary parts of a holomorphic function are *harmonic* (i.e., they both satisfy Laplace's equation).
- (iii) **(optional)** Show that each of the following functions is harmonic:
 $e^x \cos y$; $e^{x^2-y^2} \cos 2xy$; and $\ln |f(z)|$, where $f(z)$ is holomorphic.
- (b) Let the position at time t of a particle moving in the complex plane be $z(t) = r(t) \exp i\theta(t)$.
- (i) Compute the radial and transverse components of the acceleration of the particle.
- (ii) Deduce that if the particle is moving in a central force field centered at the origin then the areal speed $r^2\dot{\theta}/2$ is constant.