> [redacted] Problem 2 [redacted]

>

> $eq := diff(x(t), t) + x(t) + \varepsilon \cdot x(t)^2;$

$$eq := \frac{d}{dt} x(t) + x(t) + \varepsilon\, x(t)^2 \tag{1}$$

> **# (a) Look for solution in the form of regular perturbation expansion :**

> $s0 := x(t) = x0(t) + \varepsilon \cdot x1(t) + \varepsilon^2 \cdot x2(t);$

$$s0 := x(t) = x0(t) + \varepsilon\, x1(t) + \varepsilon^2\, x2(t) \tag{2}$$

> $eqs := series(subs(s0, eq), \text{epsilon}, 3);$

$$eqs := \mathrm{D}(x0)(t) + x0(t) + \left(\mathrm{D}(x1)(t) + x0(t)^2 + x1(t)\right)\varepsilon + \left(\mathrm{D}(x2)(t) + x2(t) + 2\, x0(t)\, x1(t)\right)\varepsilon^2 + \mathrm{O}\left(\varepsilon^3\right) \tag{3}$$

> **# *At* O(1) we find :**

> $c0 := coeff(eqs, \text{epsilon}, 0);$
> $sl := \{dsolve(\{c0, \boldsymbol{x0(0)} = 1\}, x0(t))\};$

$$c0 := \mathrm{D}(x0)(t) + x0(t)$$

$$sl := \{x0(t) = \mathrm{e}^{-t}\} \tag{4}$$

> **# *At* O(ε) we find :**

> $c1 := subs(sl, coeff(eqs, \text{epsilon}, 1));$
> $sl := sl \ \textbf{union}\ \{dsolve(\{c1, \boldsymbol{x1(0)} = 0\}, x1(t))\};$

$$c1 := \mathrm{D}(x1)(t) + \left(\mathrm{e}^{-t}\right)^2 + x1(t)$$

$$sl := \{x0(t) = \mathrm{e}^{-t}, x1(t) = \left(\mathrm{e}^{-t} - 1\right)\mathrm{e}^{-t}\} \tag{5}$$

> **# *At* O(ε^2) we find :**

> $c2 := subs(sl, coeff(eqs, \text{epsilon}, 2));$
> $sl := sl \ \textbf{union}\ \{dsolve(\{c2, \boldsymbol{x2(0)} = 0\}, x2(t))\};$

$$c2 := \mathrm{D}(x2)(t) + x2(t) + 2\left(\mathrm{e}^{-t}\right)^2\left(\mathrm{e}^{-t} - 1\right)$$

$$sl := \{x0(t) = \mathrm{e}^{-t}, x2(t) = \left(\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1\right)\mathrm{e}^{-t}, x1(t) = \left(\mathrm{e}^{-t} - 1\right)\mathrm{e}^{-t}\} \tag{6}$$

> **# So that the perturbative solution is:**

> $subs(sl, s0);$

$$x(t) = \mathrm{e}^{-t} + \varepsilon\left(\mathrm{e}^{-t} - 1\right)\mathrm{e}^{-t} + \varepsilon^2\left(\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} + 1\right)\mathrm{e}^{-t} \tag{7}$$

> **#\` \`(b) Exact solution is :**

> $sle := dsolve(\{eq, \boldsymbol{x(0)} = 1\}, x(t)) : sle;$

$$x(t) = \frac{1}{-\varepsilon + \mathrm{e}^t + \mathrm{e}^t\,\varepsilon} \tag{8}$$

> **#\` \`(c) Expanding in small ε:**

> $x(t) = simplify(series(rhs(sle), \text{epsilon}, 3));$

$$x(t) = \mathrm{e}^{-t} - \left(-1 + \mathrm{e}^t\right)\mathrm{e}^{-2t}\varepsilon + \mathrm{e}^{-3t}\left(-1 + \mathrm{e}^t\right)^2\varepsilon^2 + \mathrm{O}\left(\varepsilon^3\right) \tag{9}$$

> **# we find the same result as in (a) - check.**