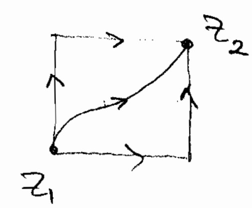
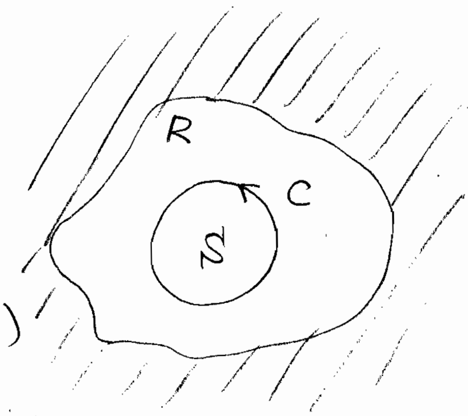
Cauchy's Integral Theorem

$$\int_{z_1}^{z_2} f(z)\,dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (u(x,y)+iv(x,y))(dx+idy)$$



Contour Integral – depends on the path.

<u>Theorem:</u> If $f(z)$-analytic in R,
then $\oint_C f(z)\,dz = 0$



$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

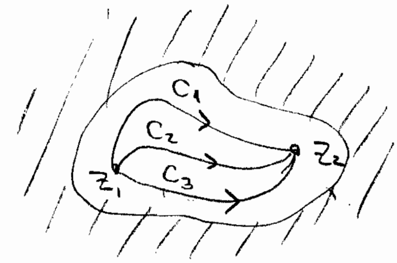
$$= -\iint_S \underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{0} dx\,dy + i\iint_S \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{0} dx\,dy = 0$$

Green's thm.

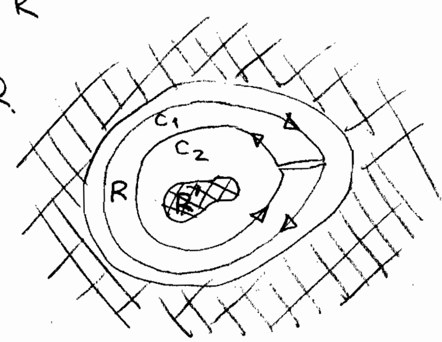
Cauchy-Riemann

<u>Corollaries:</u>

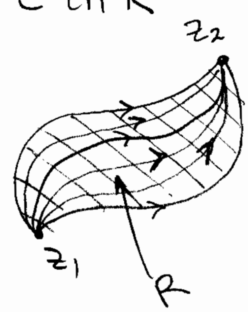
- $f(z)$-analytic in R
  $\Rightarrow$ integral is independent of path in R



- $f(z)$-analytic in non-simply-conn. R
  $\Rightarrow$ integrals <u>around</u> R' path-independ.



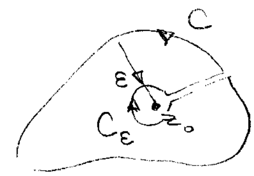
- Integral path independent $\forall c$ in R
  $\Rightarrow$ $f(z)$-analytic in R



Cauchy's Residue Theorem

What happens if the integrand is not analytic inside the loop?

$$\oint_C \frac{f(z)}{z-z_0}dz + \oint_{C_\varepsilon} \frac{f(z)}{z-z_0}dz = 0$$



$$\Rightarrow \oint_C \frac{f(z)}{z-z_0}dz = -\lim_{\varepsilon\to 0}\oint_{C_\varepsilon}\frac{f(z)}{z-z_0}dz = +\lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{f(z_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta$$

$z=z_0+\varepsilon e^{i\theta}$

$$= i f(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0)$$

$f(z)$-analytic inside C_ε

$$\Rightarrow \boxed{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), & z_0 \text{ inside } C \\ 0, & z_0 \text{ outside } C\end{cases}}$$

Derivatives:

$$f'(z_0) = \lim_{\delta z_0\to 0}\frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0} = \frac{1}{2\pi i}\lim_{\delta z_0\to 0}\left[\oint_C \frac{f(z)dz}{z-z_0-\delta z_0} - \oint_C\frac{f(z)dz}{z-z_0}\right]\frac{1}{\delta z_0} =$$

$$= \frac{1}{2\pi i}\lim_{\delta z_0\to 0}\oint_C\frac{f(z)dz}{(z-z_0-\delta z_0)(z-z_0)} = \frac{1}{2\pi i}\oint_C\frac{f(z)dz}{(z-z_0)^2}$$

and so on:
$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C\frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

Corollary:

$$\frac{1}{2\pi i}\oint_C (z-z_0)^n dz = \begin{cases} 0, & n\geq 0 \text{ (analytic)} \\ \frac{1}{2\pi i}\oint_C \frac{1}{(z-z_0)^{-n}}dz = \left(\frac{d}{dz}\right)^{(-n-1)}\frac{1}{(-n-1)!} = \begin{cases} 1, & n=-1 \\ 0, & n\leq -2\end{cases}\end{cases}$$

Series expansions:

1) $f(z)$-analytic at z_0

Taylor expansion: $f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}(z-z_0)^n f^{(n)}(z_0)$

2) $f(z)$ – non-analytic

a) z_0 – branch point (e.g., $(z-z_0)^\alpha$, α – non-integer)

$\Rightarrow$ No series expansion at z_0

b) z_0 – singular point ($f(z) \to \infty$, $z \to z_0$)

Laurent expansion: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^m} = \oint_C \sum_{n=-\infty}^{\infty} a_n (z-z_0)^{n-m}\,dz = \sum_{n=-\infty}^{\infty} a_n \oint_C (z-z_0)^{n-m}\,dz = 2\pi i\, a_{m-1}$$

$$\Rightarrow \boxed{a_m = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{m+1}}}$$

Singularities:

Simple pole: $f(z) = \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$

m^{th} order pole: $f(z) = \dfrac{a_{-m}}{(z-z_0)^m} + \ldots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$

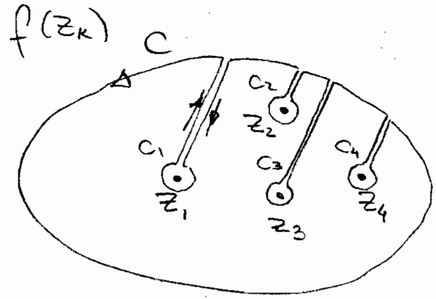
Essential singularity: $a_m \neq 0, \forall m$

Example: $f(z) = \exp\left(\frac{1}{z-z_0}\right)$

Calculus of Residues

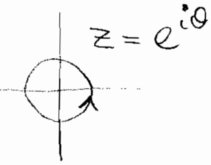
Definition: $\operatorname{res} f(z_0) \equiv a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz$ (no other poles inside C)

Residue Theorem: $\oint_C f(z)\,dz = 2\pi i \sum_{1/f(z_k)=0} \operatorname{res} f(z_k)$



Evaluation of Integrals

a) $I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, f - rational func.



$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - z^{-1})$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$

$dz = ie^{i\theta} d\theta \Rightarrow d\theta = -i\frac{1}{z}dz$

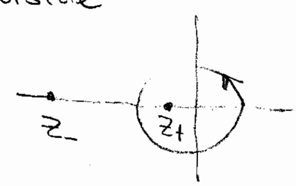
$$\Rightarrow I = -i \oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z} = 2\pi \sum_{\text{unit circle}} \text{res}\left(\frac{f(z)}{z}\right)$$

Example:

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon\cos\theta} = -i \oint \frac{dz}{z(1 + \varepsilon/2(z + z^{-1}))} = -i\frac{2}{\varepsilon}\oint \frac{dz}{z^2 + \frac{2}{\varepsilon}z + 1}$$

roots: $z_\pm = -\frac{1}{\varepsilon} \pm \frac{1}{\varepsilon}\sqrt{1 - \varepsilon^2}$, $|\varepsilon| < 1 \Rightarrow \begin{cases} z_+ - \text{inside} \\ z_- - \text{outside} \end{cases}$

$$I = -i\frac{2}{\varepsilon}2\pi i \left.\frac{1}{z + \frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1 - \varepsilon^2}}\right|_{z = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1 - \varepsilon^2}} = \frac{2\pi}{\sqrt{1 - \varepsilon^2}}$$

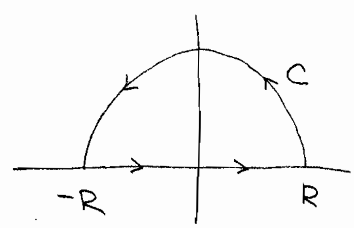


b) $I = \int_{-\infty}^{\infty} f(x)\,dx$, i) $f(z)$ - meromorphic on upper half-plane

analytic w/exception of finite # of poles

ii) $f(z) = O(|z|^{-2})$, $|z| \to \infty$

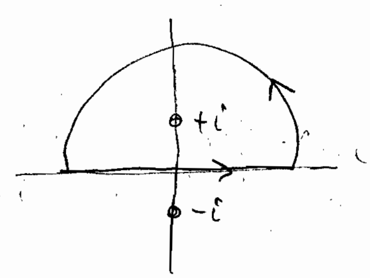
$$\oint_C = \underbrace{\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx}_{I} + \underbrace{\lim_{R\to\infty} \int_0^{\pi} f(Re^{i\theta}) Rie^{i\theta} d\theta}_{0}$$



$$\Rightarrow I = 2\pi i \sum_{\text{upper half-plane}} \text{res}(f(z))$$

Example:

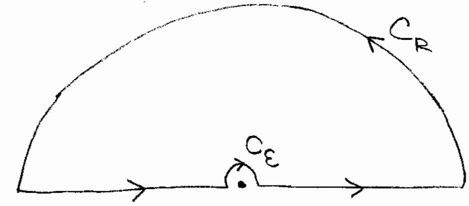
$I = \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx$: roots $z = \pm i$



$$I = 2\pi i \left.\frac{1}{z + i}\right|_{z = +i} = \frac{2\pi i}{2i} = \pi$$

c) Singularity on the contour of integration

Example: $I = \int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz$



$$\oint_C = \int_{C_\varepsilon} + \int_{C_R} + \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}$$

$$\int_{C_R} \frac{e^{iz}}{z} dz = i \int_0^\pi \underbrace{e^{iR\cos\theta}}_{|\cdot| = 1} \underbrace{e^{-R\sin\theta}}_{\to 0} d\theta \to 0$$

$$\int_{C_\varepsilon} \frac{e^{iz}}{z} dz = -i \int_0^\pi \underbrace{e^{i\varepsilon\cos\theta}}_{\to 1} \underbrace{e^{-\varepsilon\sin\theta}}_{\to 1} d\theta \to -\pi i = -\frac{1}{2}\left(2\pi i \operatorname{res} \frac{e^{iz}}{z}\Big|_{z=0}\right)$$

$$\lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{e^{iz}}{z} dz + \int_{+\varepsilon}^{+\infty} \frac{e^{iz}}{z} dz \right] \equiv P\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = +\pi i$$

Principal value

$$\Rightarrow I = \frac{1}{2} \operatorname{Im} P\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = \frac{\pi}{2}$$

Do examples

Example $I = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx$

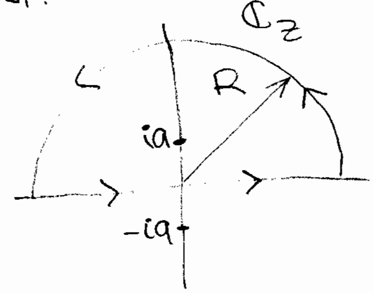
$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \to \infty$ for $|z| \to \infty$ $\Rightarrow$ cannot close contour!

Compute $I' = \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+a^2} dz$ instead ($I = \operatorname{Re} I'$)

$$z = Re^{i\theta}: \quad e^{iz} = e^{iRe^{i\theta}} = e^{iR\cos\theta - R\sin\theta} = e^{-R\sin\theta} \underbrace{e^{iR\cos\theta}}_{|\cdot| = 1},$$

$e^{-R\sin\theta} \to 0$, $R \to \infty$ for $\sin\theta > 0$, i.e., $0 < \theta < \pi$.

$\Rightarrow$ close contour in upper half-plane.



$$I' = \oint_C \frac{e^{iz} dz}{z^2+a^2} - \underbrace{\oint_{C_R} \frac{e^{iz} dz}{z^2+a^2}}_{\to 0,\ R \to \infty} =$$

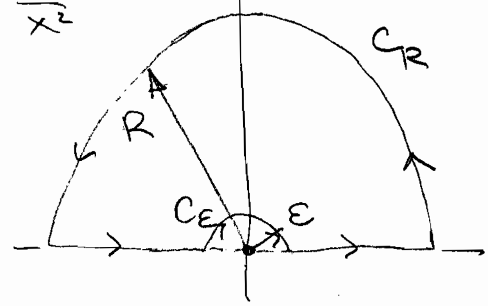
$$= 2\pi i \operatorname{res} \frac{e^{iz}}{(z-ia)(z+ia)}\Big|_{z=ia} = 2\pi i \frac{e^{-a}}{2ia} = \pi \frac{e^{-a}}{a}$$

$$I = \operatorname{Re} I' = \frac{\pi}{a} e^{-a}.$$

Example: $I = \int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2}\, dx$

1) Singularity on contour of integration: $\frac{1}{x^2}$

2) $\cos ax / \cos bx$ diverge for $|x| \to \infty$



$$I = \operatorname{Re} I' = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ibz} - e^{iaz}}{z^2}\, dz$$

$$\oint_C = \int_{C_\varepsilon} + P\int_{-\infty}^{\infty} + \int_{C_R} = 0 \quad \text{(no poles inside } c\text{)}$$

$$\Rightarrow I = \operatorname{Re} P\int_{-\infty}^{\infty} = -\operatorname{Re} \int_{C_\varepsilon} \frac{e^{ibz} - e^{iaz}}{z^2}\, dz =$$

$$= -\lim_{\varepsilon \to 0} \operatorname{Re} \int_{C_\varepsilon} \frac{(1 + ibz - \frac{1}{2}b^2z^2 + \dots) - (1 + iaz - \frac{1}{2}az^2 + \dots)}{z^2}\, dz$$

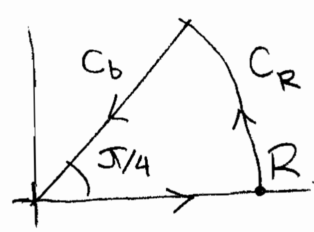
$$= -\lim_{\varepsilon \to 0} \operatorname{Re} \int_{C_\varepsilon} \Big(\frac{i(b-a)}{z} \underbrace{- \frac{1}{2}(b^2 - a^2) + \dots}_{\text{analytic}} \Big)\, dz = (a-b) \operatorname{Re} \int_{C_\varepsilon} \frac{i}{z}\, dz =$$

$$= (a-b) \operatorname{Re}\Big[\underbrace{\tfrac{1}{2}}_{\text{Half contour}} \cdot 2\pi i \underbrace{(-1)}_{\text{wrong direction}} \operatorname{res} \frac{i}{z}\Big|_{z=0} \Big] = \pi(a-b) \operatorname{Re} 1 = \pi(a-b)$$

Example: $I = \int_0^{\infty} \cos t^2\, dt = \operatorname{Re} \int_0^{\infty} e^{iz^2}\, dz$

$$C_R: \; z = Re^{i\theta} \Rightarrow e^{iz^2} = e^{iR^2 e^{2i\theta}} = e^{i(R^2\cos 2\theta + iR^2 \sin 2\theta)} =$$

$$= \underbrace{e^{iR^2\cos 2\theta}}_{1 \cdot 1 = 1} \underbrace{e^{-R^2 \sin 2\theta}}_{\to 0} \Rightarrow \sin 2\theta > 0 \Rightarrow 0 < \theta < \frac{\pi}{2}$$



Take $\theta = \frac{\pi}{4}$ on return from ∞:

$$z = \frac{1+i}{\sqrt{2}} R: \; dz = \frac{1+i}{\sqrt{2}}\, dR, \; e^{iz^2} = e^{-R^2}$$

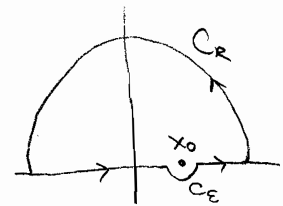
$$\oint_C e^{iz^2}\, dz = \int_0^{\infty} + \underbrace{\int_{C_R}}_{\to 0} + \int_{C_b} = 0$$

$$\Rightarrow I = \operatorname{Re} \int_0^{\infty} = -\operatorname{Re} \int_{C_b} = \operatorname{Re} \int_0^{\infty} e^{-R^2} \frac{1+i}{\sqrt{2}}\, dR = \frac{1}{\sqrt{2}} \int_0^{\infty} e^{-R^2}\, dR = \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{2}$$

Dispersion Relations (EM (optics), QM (scattering theory))

Suppose: 1) $f(z)$ - analytic in upper half-plane, and real axis and

2) $\lim\limits_{|z|\to\infty} |f(z)| = 0, \quad 0 < \arg z < \pi$



such that $f(z_0) = \dfrac{1}{2\pi i} \oint \dfrac{f(z)}{z-z_0} dz = \dfrac{1}{2\pi i} \int_{-\infty}^{\infty} \dfrac{f(x)}{x-z_0} dx$

On the real axis:

$$f(x_0) = \frac{1}{2\pi i}\Big[P\int_{-\infty}^{\infty} + \underbrace{\int_{C_\varepsilon}}_{\pi i f(x_0)} + \underbrace{\int_{C_R}}_{0}\Big] \Rightarrow f(x_0) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} dx$$

$$f(x_0) = u(x_0) + i v(x_0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)}{x-x_0} dx - \frac{i}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} dx$$

$$\Rightarrow \boxed{\begin{aligned} u(x_0) &= \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)}{x-x_0} dx \\ v(x_0) &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} dx \end{aligned}} \rightarrow$$

u, v are Hilbert transforms of each other!

Causality and Dispersion Relations

transfer function

G causes H : $H(t) = \int_{-\infty}^{\infty} F(t-t') G(t') dt'$,

$F(t-t') = 0, \quad t - t' < 0 \quad (t < t')$

Convolution: $\Rightarrow h(\omega) = f(\omega) g(\omega)$; h, f, g – Fourier transforms

Titchmarsh Theorem: If $\int_{-\infty}^{\infty} |f(\omega)|^2 d\omega < \infty$, following statements are equivalent: a) $F(t) = 0, \ t < 0$

b) $\mathrm{Re} f$, $\mathrm{Im} f$ are Hilbert transforms of each other.

Optical Dispersion

Electric permittivity $\varepsilon \neq 1$,
Magnetic permeability $\mu = 1$,
Conductivity $\sigma \neq 0$

Maxwell equations $\Rightarrow \vec{E}, \vec{B} \sim e^{ikx - i\omega t}$, $k^2 = \varepsilon \frac{\omega^2}{c^2}\left(1 + i \frac{4\pi\sigma}{\omega\varepsilon}\right)$

For poor conductivity $(\sigma \ll \omega\varepsilon)$, $k = \sqrt{\varepsilon}\frac{\omega}{c} + i \frac{2\pi\sigma}{c\sqrt{\varepsilon}}$

$$\Rightarrow e^{ikx - i\omega t} = e^{i\sqrt{\varepsilon}\frac{\omega}{c}\left(x - \underbrace{\left(\frac{1}{\sqrt{\varepsilon}}c\right)}_{\text{phase velocity}} t\right)} e^{-\overbrace{2\pi\sigma/c\sqrt{\varepsilon}}^{} x}$$

Conductive losses $\Rightarrow$ dissipation $\Rightarrow$ attenuation

Index of refraction: $n = \frac{ck}{\omega} \Rightarrow n^2 = \varepsilon + i \frac{4\pi\sigma}{\omega}$

- $n^2(\omega) \to 1$, $\omega \to \infty$ (since $\varepsilon(\omega) \to 1$ & $\sigma(\omega) \to 0$ for $\omega \to \infty$)
- $n^2(-\omega) = (n^2(\omega))^*$

$$\Rightarrow \left.\begin{aligned} \mathrm{Re}(n^2(\omega_0) - 1) &= \frac{2}{\pi} P \int_0^{\infty} \frac{\omega \, \mathrm{Im}(n^2(\omega) - 1)}{\omega^2 - \omega_0^2} d\omega \\ \mathrm{Im}(n^2(\omega_0) - 1) &= -\frac{2}{\pi} P \int_0^{\infty} \frac{\omega_0 \, \mathrm{Re}(n^2(\omega) - 1)}{\omega^2 - \omega_0^2} d\omega \end{aligned}\right\}$$ "Kronig-Kramers dispersion relations"