mathematical methods - week 11

Group theory

Georgia Tech PHYS-6124

Homework HW #11 due Monday, November 4, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 11.1 *Decompose a representation of* S_3
(a) 2; (b) 2; (c) 3; and (d) 3 points
(e) 2 and (f) 3 points bonus points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited November 2, 2019

Week 11 syllabus

Monday, October 28, 2019

Irreps, or "irreducible representations.".

Mon Harter's Sect. *3.2 First stage of non-Abelian symmetry analysis*
group multiplication table (3.1.1); class operators; class multiplication table (3.2.1b); all-commuting or central operators;

Wed Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis*
projection operators (3.2.15); 1-dimensional irreps (3.3.6); 2-dimensional irrep (3.3.7); Lagrange irreps dimensionality relation (3.3.17)

Fri Lie groups, sect. 11.3

- Definition of a Lie group
- Cyclic group $C_N \to$ continuous $SO(2)$ plane rotations
- Infinitesimal transformations, $SO(2)$ generator of rotations
- $SO(2)$ (group element) = $\exp$(generator)

11.1 It's all about class

You might want to have a look at Harter [4] *Double group theory on the half-shell* (click here). Read appendices B and C on spectral decomposition and class algebras. Article works out some interesting examples.

See also remark 1.1 *Projection operators* and perhaps watch Harter's online lecture from Harter's online course.

There is more detail than what we have time to cover here, but I find Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis* particularly illuminating. It shows how physically different (but mathematically isomorphic) higher-dimensional irreps are constructed corresponding to different subgroup embeddings. One chooses the irrep that corresponds to a particular sequence of physical symmetry breakings.

11.2 Lie groups

In week 1 we introduced projection operators (1.33). How are they related to the character projection operators constructed in the group theory lectures? While the character orthogonality might be wonderful, it is not very intuitive - it's a set of solutions to a set of symmetry-consistent orthogonality relations. You can learn a set of rules that enables you to construct a character table, but it does not tell you what it means. Similar thing will happen again when we turn to the study of continuous groups: all semisimple Lie groups will be classified by Killing and Cartan by a more complex set of orthogonality and integer-dimensionality (Diophantine) constraints. You obtain all possible Lie algebras, but have no idea what their geometrical significance is.

In my own Group Theory book [1] I (almost) get all simple Lie algebras using projection operators constructed from invariant tensors. What that means is easier to understand for finite groups, and here I like the Harter's exposition [3] best. Harter constructs 'class operators', shows that they form a basis for the algebra of 'central' or 'all-commuting' operators, and uses their characteristic equations to construct the projection operators (1.34) from the 'structure constants' of the finite group, i.e., its class multiplication tables. Expanded, these projection operators are indeed the same as the ones obtained from character orthogonality.

11.3 Continuous groups: unitary and orthogonal

Friday's lecture is not taken from any particular book, it's about basic ideas of how one goes from finite groups to the continuous ones that any physicist should know. We have worked one example out earlier, in ChaosBook *Sect. A24.4*. It gets you to the continuous Fourier transform as a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. So we need a fresh restart; that is afforded by matrix groups, and in particular the unitary group $\mathrm{U}(n) = \mathrm{U}(1) \otimes \mathrm{SU}(n)$, which contains all other compact groups, finite or continuous, as subgroups.

The main idea in a way comes from discrete groups: the whole cyclic group C_N is generated by the powers of the smallest rotation by $2\pi/N$, and in the $N \to \infty$ limit one only needs to understand the algebra of T_ℓ, generators of infinitesimal transformations, $D(\theta) = 1 + i \sum_\ell \theta_\ell T_\ell$.

These thoughts are spread over chapters of my book [1] *Group Theory - Birdtracks, Lie's, and Exceptional Groups* that you can steal from my website, but the book itself is too sophisticated for this course. If you ever want to learn some group theory in depth, you'll have to petition the School to offer it.

References

[1] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[2] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Dover, New York, 1962).

[3] W. G. Harter, *Principles of Symmetry, Dynamics, and Spectroscopy* (Wiley, New York, 1993).

[4] W. G. Harter and N. dos Santos, "Double-group theory on the half-shell and the two-level system. I. Rotation and half-integral spin states", Amer. J. Phys. **46**, 251–263 (1978).

[5] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

Exercises

11.1. **Decompose a representation of S_3.** Consider a reducible representation $D(g)$, i.e., a representation of group element g that after a suitable similarity transformation takes form

$$D(g) = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & 0 \\ 0 & D^{(b)}(g) & 0 & 0 \\ 0 & 0 & D^{(c)}(g) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix} ,$$

with character for class $\mathcal{C}$ given by

$$\chi(\mathcal{C}) = c_a\, \chi^{(a)}(\mathcal{C}) + c_b\, \chi^{(b)}(\mathcal{C}) + c_c\, \chi^{(c)}(\mathcal{C}) + \cdots ,$$

where c_a, the multiplicity of the ath irreducible representation (colloquially called "irrep"), is determined by the character orthonormality relations,

$$c_a = \overline{\chi^{(a)*}\, \chi} = \frac{1}{h} \sum_k^{class} N_k \chi^{(a)}(\mathcal{C}_k^{-1})\, \chi(\mathcal{C}_k)\,. \tag{11.1}$$

Knowing characters is all that is needed to figure out what any reducible representation decomposes into!

As an example, let's work out the reduction of the matrix representation of S_3 permutations. The identity element acting on three objects $[a\,b\,c]$ is a 3×3 identity matrix,

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing the first and second object yields $[b\ a\ c]$, represented by the matrix

$$D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

a) Find all six matrices for this representation.

b) Split this representation into its conjugacy classes.

c) Evaluate the characters $\chi(\mathcal{C}_j)$ for this representation.

d) Determine multiplicities c_a of irreps contained in this representation.

e) Construct explicitly all irreps.

f) Explain whether any irreps are missing in this decomposition, and why.

11.2. **Invariance under fractional rotations.** Argue that if the isotropy group of the velocity field $v(x)$ is the discrete subgroup C_m of SO(2) rotations about an axis (let's say the 'z-axis'),

$$C^{1/m} v(x) = v(C^{1/m} x) = v(x)\,, \qquad (C^{1/m})^m = e\,,$$

the only non-zero components of Fourier-transformed equations of motion are a_{jm} for $j = 1, 2, \cdots$. Argue that the Fourier representation is then the quotient map of the dynamics, $\mathcal{M}/\mathrm{C}_m$. (Hint: this sounds much fancier than what is - think first of how it applies to the Lorenz system and the 3-disk pinball.)

11.3. **Characters of D_3.** (continued from exercise 10.3) $\mathrm{D}_3 \cong \mathrm{C}_{3v}$, the group of symmetries of an equilateral triangle: has three irreducible representations, two one-dimensional and the other one of multiplicity 2.

(a) All finite discrete groups are isomorphic to a permutation group or one of its subgroups, and elements of the permutation group can be expressed as cycles. Express the elements of the group D_3 as cycles. For example, one of the rotations is (123), meaning that vertex 1 maps to 2, $2 \to 3$, and $3 \to 1$.

(b) Use your representation from exercise 10.3 to compute the D_3 character table.

(c) Use a more elegant method from the group-theory literature to verify your D_3 character table.

(d) Two D_3 irreducible representations are one dimensional and the third one of multiplicity 2 is formed by $[2\times2]$ matrices. Find the matrices for all six group elements in this representation.

(Hint: get yourself a good textbook, like Hamermesh [2] or Tinkham [5], and read up on classes and characters.)