mathematical methods - week 16

Calculus of variations

Georgia Tech PHYS-6124

Homework HW #16 due whenever in 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Optional exercises

Problem set #0 *Farmer and the pig*	0 points
Problem set #1 *Fermat's principle*	0 points
Problem set #2 *Lagrange multipliers*	0 points

No points for any of these, but solutions available upon request.

Perhaps, by now, you have gotten drift of what this course is about. We always start with something intuitively simple and obvious; all topics picked from current, ongoing research across physics and engineering. Then we let mathematics take over, and bang! math takes you someplace where your intuition is wrong, and the surprising path that Nature had picked instead is stunningly beautiful. Like quantum mechanics, or electron spin.

There is so much more essential mathematical physics we never got to... For example, I would have loved to discuss the calculus of variations, and whether the God does exist?

In either case, I wish you a great, stimulating, rewarding 2020!

Mon Calculus of variations.

- P. Goldbart notes.
- M. Stone & P. Goldbart Chapter 1

Wed (No class meeting) Lagrange-Euler equations; Lagrange multipliers.

- M. Stone & P. Goldbart Chapter 1
- M. Stone & P. Goldbart Section 1.5
- Read only if things nonlinear interest you: *A variational principle for periodic orbit searches*; read *Turbulent fields and their recurrences* first.

Fri (No class meeting) From Lagrange to Hamilton; Constrained Hamiltonian systems; Dirac constraints.

- Cristel Chandre *Lagrange to Hamilton* notes
- Cristel Chandre *Dirac constraints* notes

16.1 Calculus of variations

Think globally, act locally.
— Patrick Geddes

It all started by pondering elementary classical problems, such as reflection and refraction of light rays, but by the end of 18th century d'Alembert (1742), Maupertuis (1744) and Italian immigrant to France Joseph-Louis de La Grange (1788) knew that classical mechanics can be recast -in a demonstration of divine elegance- as a variational condition on a single scalar (!) function. In physics that is known as "the principle of least action," in engineering as "constrained optimization," and the function is known as respectively the "Lagrangian" or the "cost" function.

In usual physics indoctrination classes, Lagrangians are always presented as some partial derivative gymnastics relatives of the cuddly old Hamiltonians. But that totally misses the point. Hamiltonian formulation obtains solutions of natural laws by integration in time for given, *locally* specified initial conditions. Lagrangian formulation seeks *global* solutions, to be obtained without any time or space integrations.

Maupertuis saw this as a proof of the existence of God, Lagrange as merely a bag of useful mathematical tricks to solve mechanical problems with constraints. Neither could dream that in 20th century the Lagrangians would be central to the formulation of special and general relativity, quantum mechanics and quantum field theory (Feynman path integrals), a succinct statement of the myriad spatiotemporal and internal symmetries of modern particle physics.

We illustrate the principle

Example 16.1. ***Gaussian minimizes information.*** *Shannon information entropy is given by*

$$S[\rho] = -\langle \ln \rho \rangle = \int_{\mathcal{M}} dx\, \rho(x)\, \ln \rho(x)\,, \tag{16.1}$$

where ρ is a probability density. Shannon thought of $-\ln \rho$ as 'information' in the sense that if -for example- $\rho(x) = 2^{-6}$, it takes $-\ln \rho = 6$ bits of 'information' to specify the probability density ρ at the point x. Information entropy (16.1) is the expectation value of (or average) information.

A probability density $\rho \geq 0$ is an arbitrary function, of which we only require that it is normalized as a probability,

$$\int_{\mathcal{M}} dx\, \rho(x) = 1\,, \tag{16.2}$$

has a mean value,

$$\int_{\mathcal{M}} dx\, x\, \rho(x) = \mu\,, \tag{16.3}$$

and has a variance

$$\int_{\mathcal{M}} dx\, x^2 \rho(x) = \mu^2 + \sigma^2\,. \tag{16.4}$$

As ρ can be arbitrarily wild, it might take much "information" to describe it. Is there a function $\rho(x)$ that contains the least *information, i.e., that minimizes the information entropy (16.1)?*

To find it, we minimize (16.1) subject to constraints (16.2)-(16.4), implemented by adding Lagrange multipliers λ_j

$$\begin{aligned} C[\rho] &= \int_{\mathcal{M}} dx\, \rho(x)\, \ln \rho(x) \\ &\quad + \lambda_0 \left(\int_{\mathcal{M}} dx\, \rho(x) - 1 \right) + \lambda_1 \left(\int_{\mathcal{M}} dx\, x\, \rho(x) - \mu \right) \\ &\quad + \lambda_2 \left(\int_{\mathcal{M}} dx\, (x-\mu)^2 \rho(x) - \sigma^2 \right)\,, \end{aligned} \tag{16.5}$$

and looking for the extremum $\delta C = 0$,

$$\frac{\delta C[\rho]}{\delta \rho(x)} = (\ln \rho(x) + 1) + \lambda_0 + \lambda_1 x + \lambda_2 x^2 = 0\,, \tag{16.6}$$

so

$$\rho(x) = e^{-(1+\lambda_0+\lambda_1 x+\lambda_2 x^2)}\,. \tag{16.7}$$

The Lagrange multipliers λ_j can be expressed in terms of distribution parameters μ and σ by substituting this $\rho(x)$ into the constraint equations (16.2)-(16.4), and we find that the probability density that minimizes information entropy is the Gaussian

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x-\mu)^2}{2\sigma^2}} \,. \tag{16.8}$$

In what sense is that the distribution with the 'least information'? As we saw in the derivation of the cumulant expansion eq. (20.17), for a Gaussian distribution all cumulants but the mean μ and the variance σ^2 vanish, it is a distribution specified by only two 'informations', the location of its peak and its width.

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