mathematical methods - week 3

Go with the flow

Georgia Tech PHYS-6124

Homework HW #3 due Monday, September 9, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 3.1 *Rotations in a plane*	4 points
Exercise 3.2 *Visualizing 2-dimensional linear flows*	6 points

Bonus points

Exercise 3.3 *Visualizing Duffing flow*	3 points
Exercise 3.4 *Visualizing Lorenz flow*	2 points
Exercise 3.5 *A limit cycle with analytic Floquet exponent*	6 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited September 4, 2019

Week 3 syllabus

Wednesday, September 4, 2019

- Sect. 3.1 *Linear flows*
- Sect. 3.2 *Stability of linear flows*
- Optional reading: Sect. 3.3 *Nonlinear flows*
- Sect. 3.4 *Optional listening*

Typical ordinary differential equations course spends most of time teaching you how to solve linear equations, and for those our spectral decompositions are very instructive. Nonlinear differential equations (as well as the differential geometry) are much harder, but still (as we already discussed in sect. 1.3), linearizations of flows are a very powerful tool.

3.1 Linear flows

Linear is good, nonlinear is bad.
—Jean Bellissard

(Notes based of ChaosBook.org/chapters/flows.pdf)

A *dynamical system* is defined by specifying a state space $\mathcal{M}$, and a law of motion, typically an ordinary differential equation (ODE), first order in time,

$$\dot{x} = v(x)\,. \tag{3.1}$$

The vector field $v(x)$ can be any nonlinear function of x, so it pays to start with a simple example. *Linear* dynamical system is the simplest example, described by linear differential equations which can be solved explicitly, with solutions that are good for all times. The state space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the equations of motion are written in terms of a state space point x and a constant A as

$$\dot{x} = A\,x\,. \tag{3.2}$$

Solving this equation means finding the state space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through a given initial point x_0. If $x(t)$ is a solution with $x(0) = x_0$ and $y(t)$ another solution with $y(0) = y_0$, then the linear combination $ax(t) + by(t)$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + by_0$. At any instant in time, the space of solutions is a d-dimensional vector space, spanned by a basis of d linearly independent solutions.

Solution of (3.2) is given by the exponential of a constant matrix

$$x(t) \,=\, J^t\,x_0\,, \tag{3.3}$$

usually defined by its series expansion (1.7):

$$J^t = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \,, \qquad A^0 = \mathbf{1} \,, \tag{3.4}$$

and that is why we started the course by defining functions of matrices, and in particular the matrix exponential. As we discuss next, that means that depending on the eigenvalues of the matrix A, solutions of linear ordinary differential equations are either growing or shrinking exponentially (over-damped oscillators; $\cosh$'s, $\sinh$'s), or oscillating (under-damped oscillators; $\cos$'s, $\sin$'s).

3.2 Stability of linear flows

The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations (3.2) by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \,.$$

The infinitesimal deviation vector δx is thus transported along the trajectory $x(x_0, t)$, with time variation given by

$$\frac{d}{dt} \delta x_i(x_0, t) = \sum_j \frac{\partial v_i}{\partial x_j}(x) \bigg|_{x = x(x_0, t)} \delta x_j(x_0, t) \,. \tag{3.5}$$

As both the displacement and the trajectory depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0, t) \to x(t) \to x$, $\delta x_i(x_0, t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

$$\dot{x}_i = v_i(x) \,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x) \delta x_j \tag{3.6}$$

governs the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in T\mathcal{M}_x$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix* or *velocity gradients matrix*

$$A_{ij}(x) = \frac{\partial}{\partial x_j} v_i(x) \tag{3.7}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow. In case at hand, the linear flow (3.2), with $v(x) = A\,x$, the stability matrix

$$A_{ij}(x) = \frac{\partial}{\partial x_j} v_i(x) = A_{ij} \tag{3.8}$$

is a space- and time-independent constant matrix.

Consider an infinitesimal perturbation of the initial state, $x_0 + \delta x$. The perturbation $\delta x(x_0, t)$ evolves as $x(t)$ itself, so

$$\delta x(t) \;=\; J^t \, \delta x(0) \,. \tag{3.9}$$

The equations are linear, so we can integrate them. In general, the Jacobian matrix J^t is computed by integrating the *equations of variations*

$$\dot{x}_i = v_i(x) \,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x) \delta x_j \,, \tag{3.10}$$

but for linear ODEs everything is known once eigenvalues and eigenvectors of A are known.

Example 3.1. ***Linear stability of 2-dimensional flows:*** *For a 2-dimensional flow the eigenvalues λ_1, λ_2 of A are either real, leading to a linear motion along their eigenvectors, $x_j(t) = x_j(0) \exp(t\lambda_j)$, or form a complex conjugate pair $\lambda_1 = \mu + i\omega \,, \lambda_2 = \mu - i\omega$, leading to a circular or spiral motion in the $[x_1, x_2]$ plane, see example 3.2.*

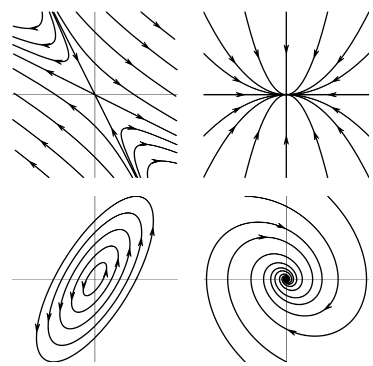

Figure 3.1: Streamlines for several typical 2-dimensional flows: saddle (hyperbolic), in node (attracting), center (elliptic), in spiral.

These two possibilities are refined further into sub-cases depending on the signs of the real part. In the case of real $\lambda_1 > 0$, $\lambda_2 < 0$, x_1 grows exponentially with time, and x_2 contracts exponentially. This behavior, called a saddle*, is sketched in figure 3.1, as are the remaining possibilities: in/out nodes, inward/outward spirals, and the center. The magnitude of out-spiral $|x(t)|$ diverges exponentially when $\mu > 0$, and in-spiral contracts into $(0, 0)$ when $\mu < 0$; whereas, the phase velocity ω controls its oscillations.*

If eigenvalues $\lambda_1 = \lambda_2 = \lambda$ are degenerate, the matrix might have two linearly independent eigenvectors, or only one eigenvector, see example 1.1. We distinguish two cases: (a) A can be brought to diagonal form and (b) A can be brought to Jordan form*, which (in dimension 2 or higher) has zeros everywhere except for the repeating eigenvalues on the diagonal and some 1's directly above it. For every such Jordan $[d_\alpha \times d_\alpha]$ block there is only one eigenvector per block.*

We sketch the full set of possibilities in figures 3.1 and 3.2.

Example 3.2. ***Complex eigenvalues: in-out spirals.*** *As $\mathbf{M}$ has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. Also the corresponding eigenvectors can be either real or complex. All coordinates used*

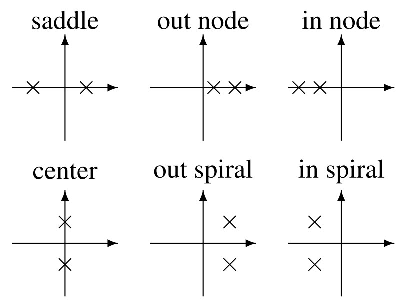

Figure 3.2: Qualitatively distinct types of exponents $\{\lambda_1, \lambda_2\}$ of a $[2\times 2]$ Jacobian matrix.

in defining a dynamical flow are real numbers, so what is the meaning of a complex *eigenvector?*

If λ_k, λ_{k+1} *eigenvalues that lie within a diagonal* $[2\times 2]$ *sub-block* $\mathbf{M}' \subset \mathbf{M}$ *form a complex conjugate pair,* $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$*, the corresponding complex eigenvectors can be replaced by their real and imaginary parts,* $\{\mathbf{e}^{(k)}, \mathbf{e}^{(k+1)}\} \to \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$*. In this 2-dimensional real representation,* $\mathbf{M}' \to A$*, the block* A *is a sum of the rescaling×identity and the generator of rotations in the* $\{\mathrm{Re}\,\mathbf{e}^{(1)}, \mathrm{Im}\,\mathbf{e}^{(1)}\}$ *plane.*

$$A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{3.11}$$

Trajectories of $\dot{\mathbf{x}} = A\,\mathbf{x}$*, given by* $\mathbf{x}(t) = J^t\,\mathbf{x}(0)$*, where (omitting* $\mathbf{e}^{(3)}, \mathbf{e}^{(4)}, \cdots$ *eigen-directions)*

$$J^t = e^{tA} = e^{t\mu} \begin{bmatrix} \cos\,\omega t & -\sin\,\omega t \\ \sin\,\omega t & \cos\,\omega t \end{bmatrix} , \tag{3.12}$$

spiral in/out around $(x, y) = (0, 0)$*, see figure 3.1, with the rotation period* T *and the radial expansion /contraction multiplier along the* $\mathbf{e}^{(j)}$ *eigen-direction per a turn of the spiral:*

$$T = 2\pi/\omega\,, \qquad \Lambda_{radial} = e^{T\mu}\,. \tag{3.13}$$

exercise 3.1

We learn that the typical turnover time scale in the neighborhood of the equilibrium $(x, y) = (0, 0)$ *is of order* $\approx T$ *(and not, let us say,* $1000\,T$*, or* $10^{-2}T$*).*

3.3 Nonlinear flows

While linear flows are prettily analyzed in terms of defining matrices and their eigenmodes, understanding nonlinear flows requires many tricks and insights. These days, we start by integrating them, by any numerical code you feel comfortable with: Matlab, Python, Mathematica, Julia, c++, whatever.

We have already made a foray into nonlinearity in example 2.2 *A simple stable/unstable manifolds pair*, but that was a bit of a cheat - it is really an example of a non-autonomous flow in variable $y(t)$, driven by external forcing by $x(t)$. Duffing

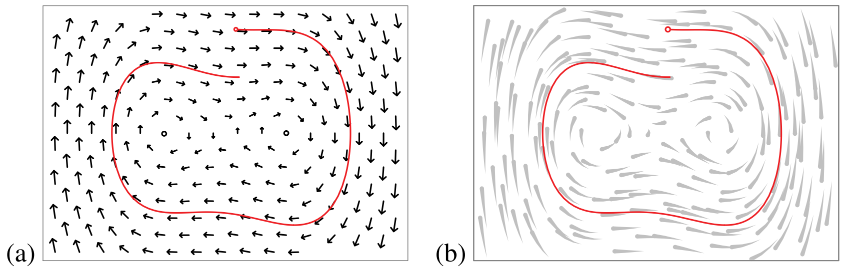

Figure 3.3: (a) The 2-dimensional vector field for the Duffing system (3.14), together with a short trajectory segment. (b) The flow lines. Each 'comet' represents the same time interval of a trajectory, starting at the tail and ending at the head. The longer the comet, the faster the flow in that region. (From ChaosBook [1])

flow of example 3.3 is a typical 2-dimensional flow, with a 'nonlinear oscialltor' limit cycle. Real fun only starts in 3 dimensions, with example 3.4 *Lorenz strange attractor*.

For purposes of this course, it would be good if you coded the next two examples, and just played with their visualizations, without further analysis (that would take us into altogether different ChaosBook.org/course1).

Example 3.3. ***A 2-dimensional vector field*** $v(x)$**.** *A simple example of a flow is afforded by the unforced Duffing system*

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -0.15\, y(t) + x(t) - x(t)^3 \end{aligned} \tag{3.14}$$

plotted in figure 3.3. The 2-dimensional velocity vectors $v(x) = (\dot{x}, \dot{y})$ *are drawn superimposed over the configuration coordinates* (x, y) *of state space* $\mathcal{M}$*.*

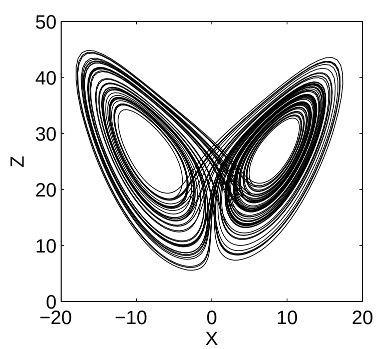

Figure 3.4: Lorenz "butterfly" strange attractor. (From ChaosBook [1])

Example 3.4. ***Lorenz strange attractor.*** *Lorenz equation*

$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{3.15}$$

has played a key role in the history of 'deterministic chaos' for many reasons that you can read about elsewhere [1]. All computations that follow will be performed for the Lorenz parameter choice $\sigma = 10, b = 8/3, \rho = 28$. For these parameter values the long-time dynamics is confined to the strange attractor depicted in figure 3.4.

3.4 Optional listening

If you do not know Emmy Noether, one of the great mathematicians of the 20th century, the time to make up for that is now. All symmetries we will use in this course are for kindergartners: flips, slides and turns. Noether, however, found a profound connections between these and invariants of our world - masses, charges, elementary particles. Then the powerful plutocrats of Germany made a clown the Chancellor of German Reich, because they could easily control him. They were wrong, and that's why you are not getting this lecture in German. Noether lost interest in physics and went on to shape much of what is today called pure mathematics.

There are no doubt many online courses vastly better presented than this one - here is a glimpse into our competition:
MIT 18.085 Computational Science and Engineering I .

References

[1] R. Mainieri, P. Cvitanović, and E. A. Spiegel, "Go with the flow", in *Chaos: Classical and Quantum*, edited by P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay (Niels Bohr Inst., Copenhagen, 2019).

Exercises

3.1. **Rotations in a plane:** In order to understand the role complex eigenvalues in example 3.2 play, it is helpful to show by exponentiation $J^t = \exp(tA) = \sum_{k=0}^{\infty} t^k A^k/k!$ with pure imaginary A in (3.11), that

$$A = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

generates a rotation in the $\{\mathrm{Re}\,\mathbf{e}^{(1)}, \mathrm{Im}\,\mathbf{e}^{(1)}\}$ plane,

$$\begin{aligned} J^t &= e^{At} = \cos\omega t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\omega t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \end{aligned} \tag{3.16}$$

3.2. **Visualizing 2-dimensional linear flows.** Either use any integration routine to integrate numerically, or plot the analytic solution of the linear flow (3.2) for all examples of qualitatively different eigenvalue pairs of figure 3.2. As noted in (1.42), the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}\,A \pm \frac{1}{2}\sqrt{(\mathrm{tr}\,A)^2 - 4\det A}$$

depend only on $\mathrm{tr}\,A$ and $\det A$, so you can get two examples by choosing any A such that $\mathrm{tr}\,A = 0$ (symplectic or Hamiltonian flow), vary $\det A$. For other examples choose A such that $\det A = 1$, vary $\mathrm{tr}\,A$. Do your plots capture the qualitative features of the examples of figure 3.1?

3.3. **Visualizing Duffing flow.** Use any integration routine to integrate numerically the Duffing flow (3.14). Take a grid of initial points, integrate each for some short time δt. Does your result look like the vector field of figure 3.3? What does a generic long-time trajectory look like?

3.4. **Visualizing Lorenz flow.** Use any integration routine to integrate numerically the Lorenz flow (3.15). Does your result look like the 'strange attractor' of figure 3.4?

3.5. **A limit cycle with analytic Floquet exponent.** There are only two examples of nonlinear flows for which the Floquet multipliers can be evaluated analytically. Both are cheats. One example is the 2-dimensional flow

$$\begin{aligned} \dot{q} &= p + q(1 - q^2 - p^2) \\ \dot{p} &= -q + p(1 - q^2 - p^2)\,. \end{aligned}$$

Determine all periodic solutions of this flow, and determine analytically their Floquet exponents. Hint: go to polar coordinates $(q, p) = (r\cos\theta, r\sin\theta)$. G. Bard Ermentrout