FRACTALS IN PHYSICS
L. Pietronero, E. Tosatti (editors)

CIRCLE MAPS IN THE COMPLEX PLANE

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Circle maps of polynomial, exponential, and rational polynomial types are studied numerically in the complex plane. The golden mean universality for real circle maps does not extend into the complex plane.

1. INTRODUCTION

The discovery of the period-doubling universality in one-dimensional iterations[1],[2] has prompted a search for universal scalings in other low dimensional dynamical systems (the theory and the experimental observations of period doublings are reviewed in refs. [3],[4],[5]). One class of such problems in which the universality ideas have had some success are the transitions to chaos for diffeomorphisms on the circle (circle maps). Maps of this type model a variety of physical systems: we refer the reader to refs. [6],[7] for a discussion of their physical applications.

The first example of a universal scaling for the critical circle maps was discovered in a study of mappings with the golden mean winding number[8],[9],[10]. An elegant formulation of such asymptotically universal self-similarities is afforded by the unstable manifold equations[11],[12],[13],[14],[3].

For the circle maps the unstable manifold equation is given by (see for example refs. [12],[15]):

$$g_p(z) = \alpha g_{1+p/\delta}(\alpha g_{1+1/\delta+p/\delta^2}(z/\alpha^2)) \qquad (1.1)$$

However, numerical solutions of this equation are made difficult by subtle convergence problems. To best of our knowledge, only two successful numerical solutions of the unstable manifold equations are extant[16]. These convergence problems, as well as the interest in understanding other universalities associated with the circle maps[15], has motivated us to investigate the structure of the complexified circle maps. Such investigations have previously yielded new insights into universal scaling laws[13],[14],[17],[18], as well as much beautiful mathematics (see for example refs. [19],[20],[21],[22],[23],[24]).

The reader is referred to refs. [6], [15] and [25] for an introduction to the circle maps. In this note we concentrate only on some general properties of complexified circle maps.

2. THE CUBIC CIRCLE MAP

As the first example of a complexified circle map, consider the critical cubic map

$$z_{n+1} = \Omega + 4z_n^3, \quad \mathrm{Re}(z + 1/2) \bmod 1 \qquad (2.1)$$

(a cubic circle map is critical if it has a cubic inflection point $z'=z''=0$). This map is periodic along the real axis, and discontinuous at $z = \pm 1/2 + iy$ for all $y \neq 0$. This is the crudest example of the polynomial approximations to circle maps of kind used in the numerical solutions of the unstable manifold equation (1.1).

The Mandelbrot set (the set of all parameter values for which the iterates of the critical point tend to infinity) for a polynomial cubic map is plotted in fig. 2.1.

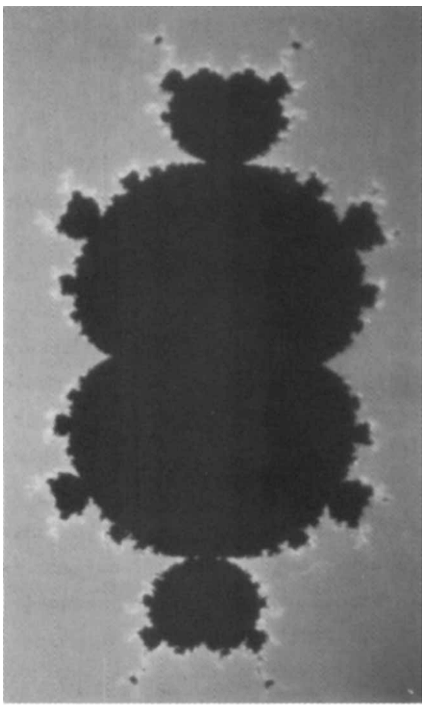

FIGURE 2.1 The Mandelbrot set for the critical polynomial cubic mapping $z \to c + 4z^3$.

The boundary of the large central component (parameter values for which the iterates of the critical point converge to a stable fixed point) is given by the cardioid of parameter values for which the fixed point is marginally stable, $|dz'/dz| = 1$. The z map is monotone for z real and therefore cannot exhibit period doubling along the real axis; however, bifurcations to period 2 occur on the imaginary axis, generating the pair of 2-cycle "hearts". One of these

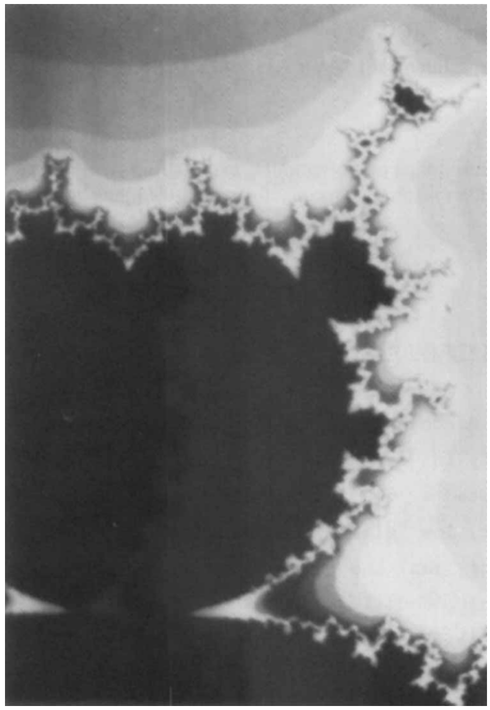

FIGURE 2.2 An enlargement of the cycle-2 "heart" from fig. 2.1.

hearts is plotted in fig. 2.2. A bifurcation to a stable periodic orbit takes place at every rational value of dz'/dz phase along the cardioid boundary. This gives rise to a set composed of self-similar "hearts", analogous to the Mandelbrot set for the quadratic map $z \to c+z^2$. Just as in the quadratic case[14], each component is centered on a superstable m/n cycle, and one expects universal scaling laws for the infinite sequences of bifurcations.

The shape of the Mandelbrot set for the circle maps will be described in the next section.

The basin of attraction (the set of all initial values of z whose iterates do not escape to infinity) of the superstable fixed point for the cubic circle map (2.1) is plotted in fig. 2.3.

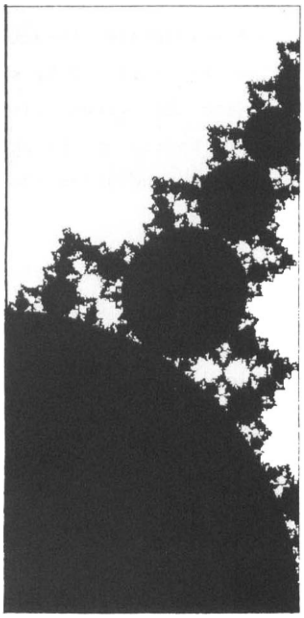

FIGURE 2.3 The upper right quadrant of the basin of attraction of the superstable fixed point for the cubic map (2.1) ($\Omega = 0$). The remaining quadrants are obtained by reflection about the x and y axes.

The general shape of the basins of attraction for the complex circle maps is easy to understand. The map (2.1) has a boundary of marginal stability $|dz'/dz| = 1$ on the circle of radius $|z| = 1/2$. This defines the central disk. The periodicity condition (Re(z+1/2) mod 1) generates an infinity of such disks, one for every integer value of Re z. The remaining self-similar structure arises from this periodicity; it is generated by all preimages of the $|z| = 1/2$ disks along the real axis. z^3 maps initial points with phase $\exp(\pm 2i\pi/3)$, $\exp(\pm i\pi/3)$ onto the real axis. This gives rise to the four rays of smaller disks at $\pm 60°$, $\pm 120°$. The higher iterates of (2.1) image similarly the other sequences of disks visible in fig. 2.3 onto the real axis. (The abrupt truncation close to Im z = .95 is an artifact of the Re z modularity condition in (2.1).)

3. THE STANDARD CIRCLE MAP

As the second example of a complexified circle map we take the sine map

$$z_{n+1} = \Omega + z_n - \frac{k}{2\pi}\sin(2\pi z_n) \ , \quad \text{mod } 1 \qquad (3.1)$$

(k=1 for the critical case). The basins of attraction have the same basic structure as in the previous example; the superstable fixed point basin of attraction is given in fig. 3.1, and the superstable 3-cycle basin of attraction is given in fig. 3.2 as typical examples.

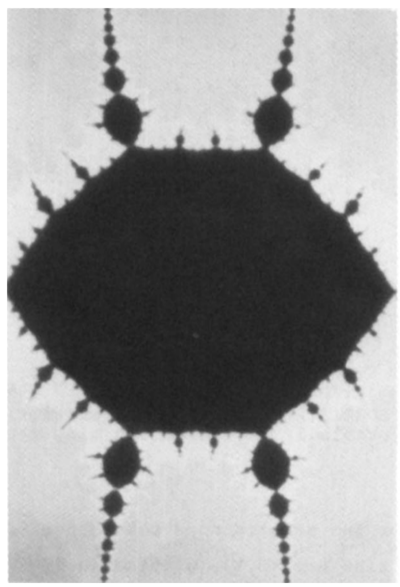

FIGURE 3.1 The basin of attraction for the superstable fixed point of the sine map (3.1).

The complexified sine map is an example of an exponential map[26]: with substitutions

$$u = \exp(2\pi i z) \qquad c = \exp(2\pi i\Omega)$$

(3.1) becomes

$$u' = cue^{-k(u-1/u)/2} \tag{3.2}$$

(here k=1). Unlike the polynomial maps of type (2.1), or the rational maps which we shall discuss in the next section, the exponential maps have basins of attraction which extend to infinity and are dense everywhere in the complex plane. For example, the image of $z = \pm 1/4 + iy$, y large, lies close to the real axis in fig. 3.2, and in the same way any ray of disks which is a preimage of the real axis extends to infinity.

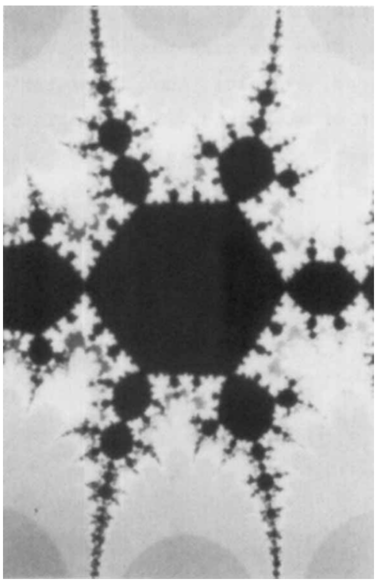

FIGURE 3.2 The basin of attraction for the superstable 1/3 cycle of the sine map (3.1).

For the same reasons the Mandelbrot set for the sine map (3.1), plotted in fig. 3.3, is dense everywhere over the entire complex plane.

The general structure of the Mandelbrot sets for the critical circle maps is illustrated by fig. 3.3 . Along the real axis there is a mode-locking interval for every rational number (the "devil's staircase"[6]). As the map has a cubic inflection, each mode-locking interval extends in the complex plane into a "hearts" set of fig. 2.1. By crossing from the central component into one of the hearts, one drives the mapping through a period n-tupling without changing its winding number (for example, the 1/2 winding number period-doubles to 2/4). Hence the Mandelbrot set for a circle map is constructed from two basic building blocks: first, the map has a cubic inflection, so the complex period n-tuplings are characterized by the self-similar hearts of fig. 2.1; second, the map is periodic, and that gives rise to the infinite sequences of copies of the basic hearts set along the real axis, and along the rays in the complex plane.

Similarly, the basins of attraction for the critical circle maps are constructed from the corresponding building blocks: first, a basin of attraction for the critical cubic

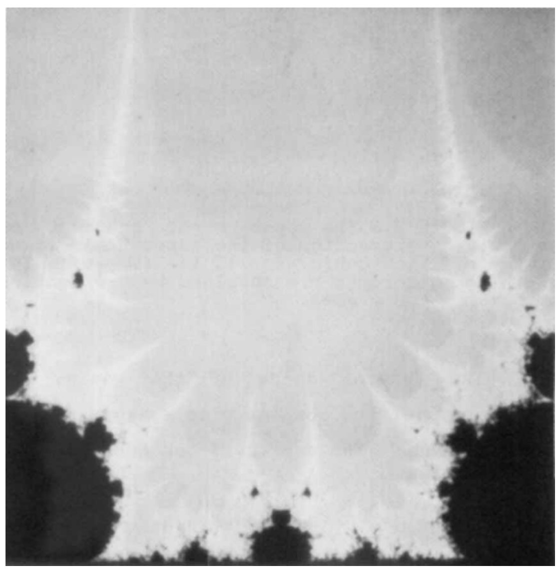

FIGURE 3.3 The Mandelbrot set for the sine map (3.1). The lower complex halfplane is obtained by the reflection about the real axis. The black region corresponds to values of parameter for which the critical point does not iterate away to infinity. The different shades of gray are an indication of the escape velocity; the longer the number of iterations needed to reach a cutoff, the lighter the shading. The large "hearts" set on the ends of the interval corresponds to the fixed point, the hearts set in the middle corresponds to the orbits with the winding number 1/2, and so on.

polynomial map; second, from the infinity of its preimages generated by the periodicity condition. This is illustrated in fig. 3.4 for the superstable period-doubled cycle with winding number 0/1. The triangular arrangement of the central 4 disks is typical of the basins of attraction for period-2 polynomial cubic maps; the remainder of the set is generated by the periodicity along the real axis, and its preimages in the complex plane.

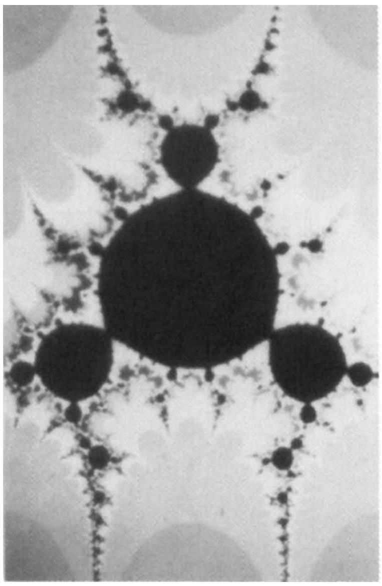

FIGURE 3.4 The basin of attraction for the superstable period-doubled cycle of (3.1) with winding number 0/1. The central part of the basin of attraction is the same as for the period-2 polynomial cubic maps. The remainder of the set are replicas generated by the periodicity along the real axis, and the preimages in the complex plane.

4. A RATIONAL POLYNOMIAL MAP

As the third example of a circle map we take the rational polynomial map

$$u' = cu(3-ku)/(3-k/u) \tag{4.1}$$

(k=1 here). The form of this map is motivated by the lowest order truncation of the power series expansion of the exponentials in (3.2). It is a critical circle map with a cubic inflection point at u=1:

$$\begin{aligned} u' &\approx c(1 - r^3/2 - r^5/4 + r^6/8 + \dots) \\ r &= 2\pi i z \end{aligned} \tag{4.2}$$

The Mandelbrot set for this map is given in fig. 4.1.

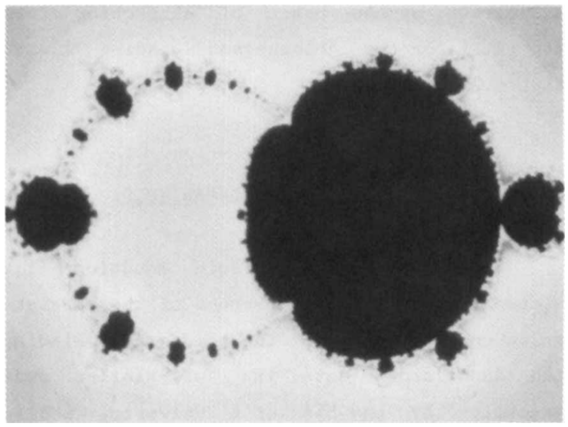

FIGURE 4.1 The Mandelbrot set for the rational map (4.1). The fixed point hearts set is centered on c=1, the 1/2 winding on c=-1. The remaining infinity of other mode lockings lie on the unit circle.

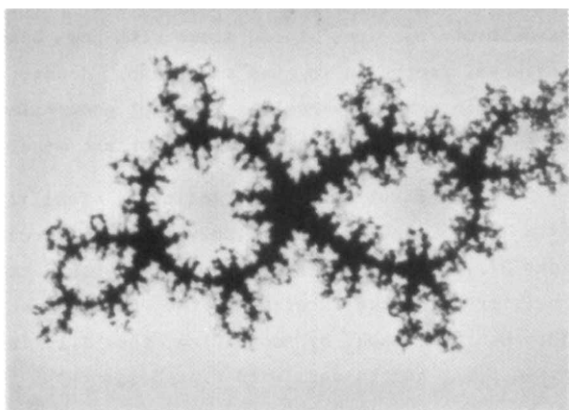

FIGURE 4.2 The basin of attraction of the rational polynomial map (4.1) for the parameter value corresponding to the golden-mean winding number.

The rational maps differ from the exponential maps in one important aspect; their Mandelbrot sets and their basins of attraction are finite in extent, because for large $|u|$ the rational maps behave like polynomials. The preimages of the real axis of the circle map (more precisely, as (4.1) is an exponentiated circle map, the preimages of the $|u|=1$ unit circle) are themselves closed loops. This is illustrated by the basin of attraction of the map (4.1) for the golden-mean winding number, fig. 4.2.

5. THE GOLDEN MEAN UNSTABLE MANIFOLD

The unstable manifold equation (1.1) states that in the neighborhood of the parameter value corresponding to the golden-mean winding, the Mandelbrot set is self-similar under rescaling by the Shenker's universal scaling number δ. We have investigated this numerically by blowing up the neighborhood of the parameter value corresponding to the golden mean winding, and comparing the sizes of the hearts sets corresponding to the successive ratios of Fibonacci numbers. In this neighborhood the Mandelbrot set does indeed scale with the same universal factor as in the real case. However, the golden mean universality does not generalize to the complex plane in several important ways.

The first non-universality is familiar from the theory of polynomial iterations[27],[24]; the longer the cycle, the "hairier" is the exterior of the hearts set. This can be seen by comparing fig. 2.1, the fixed point hearts set, with fig. 5.1.

The other non-universal feature of the complex circle maps is the fact that the exteriors of hearts sets for different maps remain unmistakably distinguishable, regardless of the degree of magnification of the the golden

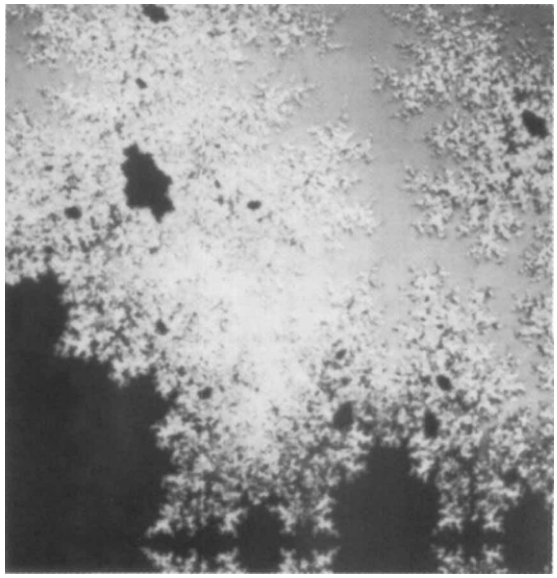

FIGURE 5.1 The Mandelbrot set fig. 3.3 f or the sine map (3.1) in the neighborhood of the parameter value corresponding to the golden mean winding. The largest hearts set corresponds to the 55/89 Fibonacci numbers ratio. While the successive Fibonacci ratios' hearts sets do scale by Shenker's universal scaling, their exteriors contain more and more "hair".

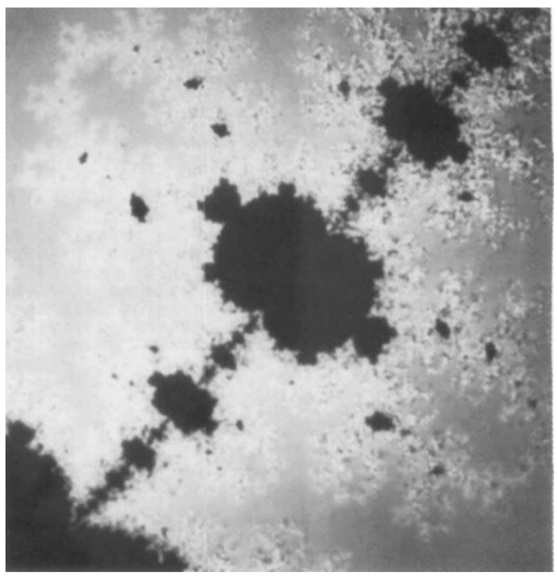

FIGURE 5.2 The Mandelbrot set fig. 4.1 for the rational polynomial map (4.1) in the neighborhood of the parameter value corresponding to the golden mean winding. The largest hearts set corresponds to the 21/34 Fibonacci numbers ratio. While the successive Fibonacci ratios' hearts sets do scale by Shenker's universal scaling, their exteriors are different from those for the exponential maps, such as fig. 5.1.

mean winding neighborhood. This can be seen by comparing fig. 5.1 with fig. 5.2. While all the external rays for the sine map go to infinity, the exteriors of hearts sets for the rational polynomial map are decorated by looplike preimages of the unit circle. Numerically, these loop decorations scale by the same universal Shenker's number as the interiors of the hearts sets, so we can always determine the type of the starting approximation to the unstable manifold, regardless of the degree of magnification. We conclude that the unstable manifold equation (1.1) has no unique analytic continuation into the complex plane.

ACKNOWLEDGEMENTS

We are grateful to J.H. Hubbard and B. Branner for many inspiring discussions. P.C. thanks J.H. Hubbard and the Department of Mathematics, Cornell University, for the access to the Mathematics VAX, and B. Wittner for the kind assistance with the two-dimensional iteration programs, with which most of the graphics in this paper were generated. P.C. is grateful to M.J. Feigenbaum for the hospitality at the Laboratory of Solid State Physics, and acknowledges support from DOE contract no. DE-AC02-83-ER13044. I.P. acknowledges partial support by the Minerva Foundation, Munich, Germany. M.H.J. and L.K. work has been supported in part by the U.S. Office of Naval Research and by the Materials Research Laboratory.

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