How well can one resolve the state space of a chaotic flow?

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All physical systems are affected by some noise that limits the resolution that can be attained in partitioning their state space. For chaotic, locally hyperbolic flows, this resolution depends on the interplay of the local stretching/contraction and the smearing due to noise. We propose to determine the 'finest attainable' partition for a given hyperbolic dynamical system and a given weak additive white noise, by computing the local eigenfunctions of the adjoint Fokker-Planck operator along each periodic point, and using overlaps of their widths as the criterion for an optimal partition. The Fokker-Planck evolution is then represented by a finite transition graph, whose spectral determinant yields time averages of dynamical observables. Numerical tests of such 'optimal partition' of a one-dimensional repeller support our hypothesis.

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The effect of noise on the behavior of a nonlinear dynamical system is a fundamental problem in many areas of science [1–3], and the interplay of noise and chaotic dynamics is of particular current interest [4–6].

The purpose of this letter is two-fold. First, and conceptually the most important, we point out an effect of noise that has not been addressed in literature: weak noise limits the attainable resolution of the state space ('phase space') of a chaotic system. We formulate the 'optimal partition' hypothesis whose implementation requires only integration of a small set of solutions of the deterministic equations of motion. Second, more technical point; we show that the optimal partition hypothesis replaces the Fokker-Planck PDEs by finite, low-dimensional Fokker-Planck matrices, whose eigenvalues give good estimates of long-time observables (escape rates, Lyapunov exponents, *etc.*).

A chaotic trajectory explores a strange attractor, and evaluation of long-time averages requires effective partitioning of the state space into smaller regions. The set of unstable periodic orbits forms a 'skeleton' that can be used to partition the state space into such smaller regions, each region a neighborhood of a periodic point [7, 8] (i.e., a point on a periodic orbit). The number of periodic orbits grows exponentially with period length, yielding finer and finer partitions, with the neighborhood of each periodic orbit shrinking exponentially.

As there is an infinity of periodic orbits, with each neighborhood shrinking asymptotically to a point, a deterministic chaotic system can - in principle - be resolved arbitrarily finely. However, any physical system suffers background noise, any numerical prediction suffers computational roundoff noise, and any set of equations models nature up to a given accuracy, since degrees of freedom are always neglected. If the noise is weak, the short-time dynamics is not altered significantly: short periodic orbits of the deterministic flow still partition coarsely the state space. Intuitively, the noise smears out the neighborhood of a periodic point, whose size is now determined by the interplay between the diffusive spreading parameterized [9, 10] by the diffusion constant D, and its exponentially shrinking deterministic neighborhood. As the periods of periodic orbits increase, the diffusion always wins, and successive refinements of a deterministic partition of the state space stop at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition. The smearing width differs from trajectory to trajectory, so there is no one single time beyond which noise takes over; rather, as we shall show here, the optimal partition has to be computed for a given dynamical system and given noise. This effort brings a handsome practical reward: as the optimal partition is finite, the Fokker-Planck operator can be represented by a finite matrix.

While the general idea is intuitive, nonlinear dynamics interacts with noise in a nonlinear way, and methods for implementing the optimal partition for a given noise still need to be developed. In this letter we propose a new approach to this partitioning. We compute the width of the leading eigenfunction of the linearized adjoint Fokker-Planck operator on each periodic point. The optimal partition is then obtained by tracking the diffusive widths of unstable periodic orbits until they start to overlap. We describe here the approach as applied to $1\,d$ expanding maps; higher-dimensional hyperbolic maps and flows require a separate treatment for contracting directions, a topic for a future publication [11].

As the simplest application of the method, consider the orbit $\{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$ of a $1\,d$ map $x_{n+1} = f(x_n)$, and the associated discrete Langevin equation [3]

$$x_{n+1} = f(x_n) + \xi_n\,, \tag{1}$$

where the ξ_n are independent Gaussian random variables of mean 0 and variance $2D$ (the method can be applied to continuous time flows as well, but a $1\,d$ map suffices to illustrate the optimal partition algorithm). The corresponding Fokker-Planck operator [1],

$$\mathcal{L} \circ \rho_n(y) = \int \frac{dx}{\sqrt{4\pi D}} \, e^{-\frac{(y-f(x))^2}{4D}} \rho_n(x) \tag{2}$$

carries the density of Langevin trajectories $\rho_n(x)$ forward in time to $\rho_{n+1} = \mathcal{L} \circ \rho_n$. Since a density concentrated at point x_n is carried into a density concentrated at x_{n+1}, we introduce local coordinate systems z_a centered on the orbit points x_a, together with a notation for the map (1), its derivative, and, by the chain rule, the derivative of the kth iterate f^k evaluated at the point x_a,

$$\begin{aligned} x &= x_a + z_a \,, \quad f_a(z_a) = f(x_a + z_a) \\ f'_a &= f'(x_a) \,, \;\; f^{k\prime}_a = f'_{a+k-1} \cdots f'_{a+1} f'_a \,, \;\; k \geq 2 \,. \end{aligned} \tag{3}$$

Here a is the label of point x_a, and the label $a+1$ is a shorthand for the next point b on the orbit of x_a, $x_b = x_{a+1} = f(x_a)$. For example, a period-3 periodic point might have label $a = 001$, and by $x_{010} = f(x_{001})$ the next point label is $b = 010$.

If the noise is weak, we can approximate (to leading order in D) the Fokker-Planck operator, $\mathcal{L}_a \circ \rho_n(x_{a+1} + z_{a+1}) = \int dz_a \mathcal{L}_a(z_{a+1}, z_a)\rho_n(x_a + z_a)$, by linearization centered on x_a, the ath point along the orbit,

$$\mathcal{L}_a(z_{a+1}, z_a) \;=\; (4\pi D)^{-1/2} \, e^{-\frac{(z_{a+1} - f'_a z_a)^2}{4D}} \,. \tag{4}$$

$\mathcal{L}_a$ maps a Gaussian density $\rho_n(x_a + z_a) = c_a \exp\left\{-z_a^2/2\sigma_a^2\right\}$, of variance σ_a^2, into a Gaussian density $\rho_{n+1}(x)$ of variance $\sigma_{a+1}^2 = (f'_a \sigma_a)^2 + 2D$. This variance is an interplay of the Brownian noise contribution $2D$ and the nonlinear contracting/amplifying contribution $(\sigma f')^2$. The diffusive dynamics of a nonlinear system are thus fundamentally different from Brownian motion, as the map induces a history dependent effective noise.

In order to determine the smallest noise-resolvable state space partition along the trajectory of x_a, we need to determine the effect of noise on the points preceding x_a. This is achieved by the *adjoint Fokker-Planck operator*

$$\mathcal{L}^\dagger \circ \tilde{\rho}_n(x) = \int \frac{dy}{\sqrt{4\pi D}} \, e^{-\frac{(y-f(x))^2}{4D}} \tilde{\rho}_n(y) \,, \tag{5}$$

which relates a density $\tilde{\rho}_n$ concentrated around x_a to $\tilde{\rho}_{n-1} = \mathcal{L}^\dagger \circ \tilde{\rho}_n$, a density concentrated around the previous point x_{a-1}, the variance transforming as $(f'_{a-1}\sigma_{a-1})^2 = \sigma_a^2 + 2D$. For an unstable (expanding) map, these variances shrink. After n steps the variance is given by

$$(f^{n\prime}_{a-n}\sigma_{a-n})^2 = \sigma_a^2 + 2D(1 + (f'_{a-1})^2 + \cdots + (f^{n-1\prime}_{a-n+1})^2) \,. \tag{6}$$

From the dynamical point of view, a good state space partition encodes the recurrent dynamics; here we shall seek a partition in terms of neighborhoods of periodic points [8, 12] of short periods. For the linearized $\mathcal{L}_a^\dagger$ acting on a fixed point $x_a = f(x_a)$, the $n \to \infty$ sum (6) converges to a Gaussian of variance

$$\sigma_a^2 = 2D/(\Lambda_a^2 - 1) \,, \tag{7}$$

where $\Lambda_a = f'_a$, and for a periodic point $x_a \in p$ to a Gaussian of variance

$$\sigma_a^2 = \frac{2D}{1 - \Lambda_p^{-2}} \left(\frac{1}{(f'_a)^2} + \cdots + \frac{1}{\Lambda_p^2} \right) , \tag{8}$$

where $\Lambda_p = f^{n_p\prime}_a$ is the Floquet multiplier (eigenvalue of the Jacobian linearized flow) of an unstable ($|\Lambda_p| > 1$) periodic orbit p of period n_p. This is the key formula; note that its evaluation requires no Fokker-Planck formalism, it depends only on the deterministic orbit and its linear stability.

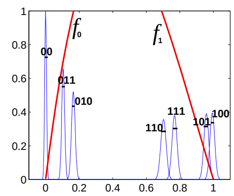

FIG. 1: f_0, f_1: branches of the deterministic map (9) for $\Lambda_0 = 8$ and $b = 0.6$. The local eigenfunctions $\tilde{\rho}_{a,0}$ with variances given by (8) provide a state space partitioning by neighborhoods of periodic points of period 3. These are computed for noise variance (D = diffusion constant) $2D = 0.002$. The neighborhoods $\mathcal{M}_{000}$ and $\mathcal{M}_{001}$ already overlap, so $\mathcal{M}_{00}$ cannot be resolved further. For periodic points of period 4, only $\mathcal{M}_{011}$ can be resolved further, into $\mathcal{M}_{0110}$ and $\mathcal{M}_{0111}$.

We can now state the main result of this letter, *'the best possible of all partitions'* hypothesis, as an algorithm: assign to each periodic point x_a a neighborhood of finite width $[x_a - \sigma_a, x_a + \sigma_a]$. Consider periodic orbits of increasing period n_p, and *stop the process of refining* the state space partition as soon as the adjacent neighborhoods overlap.

As a concrete application to the Langevin map (1) consider map [13]

$$f(x) = \Lambda_0 x(1-x)(1-bx) \tag{9}$$

plotted in figure 1; this figure also shows the local eigenfunctions $\tilde{\rho}_{a,0}$ with variances given by (8). Each Gaussian is labeled by the $\{f_0, f_1\}$ branches visitation sequence of the corresponding deterministic periodic point (a symbolic dynamics, however, is not a prerequisite for implementing the method). We find that in this case the state space (the unit interval) can be resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\} \,. \tag{10}$$

Evolution in time maps the optimal partition interval $\mathcal{M}_{011} \to \{\mathcal{M}_{110}, \mathcal{M}_{111}\}$, $\mathcal{M}_{00} \to \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}$,

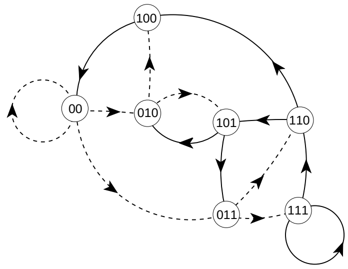

FIG. 2: Transition graph (graph whose links correspond to the nonzero elements of a transition matrix T_{ba}) describes which regions b can be reached from the region a in one time step. The 7 nodes correspond to the 7 regions of the optimal partition (10). Dotted links correspond to symbol 0, and the full ones to 1, indicating that the next region is reached by the f_0, respectively f_1 branch of the map plotted in figure 1.

etc., as compactly summarized by the transition graph of figure 2.

Next we show that the optimal partition enables us to replace Fokker-Planck PDEs by finite-dimensional matrices. The variance (8) is stationary under the action of $\mathcal{L}_a^{\dagger n_p}$, and the corresponding Gaussian is thus an eigenfunction. Indeed, for the linearized flow the entire eigenspectrum is available analytically, and will be a key ingredient in what follows. For a periodic point $x_a \in p$, the n_pth iterate $\mathcal{L}_a^{n_p}$ of the linearization (4) is the discrete time version of the Ornstein-Uhlenbeck process [14], with left $\tilde{\rho}_0, \tilde{\rho}_1, \cdots$, respectively right $\rho_0, \rho_1, \cdots$ mutually orthogonal eigenfunctions [1] given by

$$\begin{aligned} \tilde{\rho}_{a,k}(z) &= \frac{\beta^{k+1}}{\sqrt{\pi} 2^k k!} H_k(\beta z) e^{-(\beta z)^2} \\ \rho_{a,k}(z) &= \frac{1}{\beta^k} H_k(\beta z) \,, \end{aligned} \qquad (11)$$

where $H_k(x)$ is the kth Hermite polynomial, $1/\beta = \sqrt{2}\sigma_a$, and the kth eigenvalue is $1/|\Lambda|\Lambda^k$.

Partition (10) being the finest possible partition, the Fokker-Planck operator now acts as $[7\times 7]$ matrix with non-zero $a \to b$ entries expanded in the Hermite basis,

$$\begin{aligned} [\mathbf{L}_{ba}]_{kj} &= \langle \tilde{\rho}_{b,k} | \mathcal{L} | \rho_{a,j} \rangle \\ &= \int \frac{dz_b dz_a \, \beta}{2^{j+1} j! \pi \sqrt{D}} e^{-(\beta z_b)^2 - \frac{(z_b - f_a(z_a))^2}{4D}} \\ &\qquad \times H_k(\beta z_b) H_j(\beta z_a) \,, \end{aligned} \qquad (12)$$

where $1/\beta = \sqrt{2}\sigma_a$, and z_a is the deviation from the periodic point x_a. It is the number of resolved periodic points that determines the dimensionality of the Fokker-Planck matrix.

Periodic orbit theory [12, 15] expresses the long-time dynamical averages, such as Lyapunov exponents, escape rates, and correlations, in terms of the leading eigenvalues of the Fokker-Planck operator $\mathcal{L}$. In our 'optimal partition' approach, $\mathcal{L}$ is approximated by the finite-dimensional matrix $\mathbf{L}$, and its eigenvalues are determined from the zeros of $\det(1 - z\mathbf{L})$, expanded as a polynomial in z, with coefficients given by traces of powers of $\mathbf{L}$. As the trace of the nth iterate of the Fokker-Planck operator $\mathcal{L}^n$ is concentrated on periodic points $f^n(x_a) = x_a$, we evaluate the contribution of periodic orbit p to $\mathrm{tr}\,\mathbf{L}^{n_p}$ by centering $\mathbf{L}$ on the periodic orbit,

$$t_p = \mathrm{tr}_{\,p}\, \mathcal{L}^{n_p} = \mathrm{tr}\, \mathbf{L}_{ad} \cdots \mathbf{L}_{cb} \mathbf{L}_{ba} \,, \qquad (13)$$

where $x_a, x_b, \cdots x_d \in p$ are successive periodic points.

To leading order in the noise variance $2D$, t_p takes the deterministic value $t_p = 1/|\Lambda_p - 1|$. The nonlinear diffusive effects in (12) are accounted for by the weak-noise Taylor series expansion around the periodic point x_a,

$$\begin{aligned} e^{-\frac{(z_b - f_a(z_a))^2}{4D}} &= e^{-\frac{(z_b - f'_a z_a)^2}{4D}} \times \\ &\left(1 - 2\sqrt{D}(f''_a f'_a z_a^3 + f''_a z_a^2 z_b) + O(D)\right) . \end{aligned} \qquad (14)$$

Such higher order corrections will be needed in what follows for a sufficiently accurate comparison of different methods.

We illustrate the method by calculating the escape rate $\gamma = -\ln z_0$, where z_0^{-1} is the leading eigenvalue of Fokker-Planck operator $\mathcal{L}$, for the repeller plotted in figure 1. The spectral determinant can be read off the transition graph of figure 2,

$$\begin{aligned} \det(1 - z\mathbf{L}) &= 1 - (t_0 + t_1) z - (t_{01} - t_0 t_1)\, z^2 \\ &- (t_{001} + t_{011} - t_{01} t_0 - t_{01} t_1)\, z^3 - \cdots \\ &- (t_{0010111} + t_{0011101} - \cdots + t_{001} t_{011} t_1)\, z^7 . \end{aligned} \qquad (15)$$

The polynomial coefficients are given by products of non-intersecting loops of the transition graph [12], with the escape rate given by the leading root z_0^{-1} of the polynomial. Twelve periodic orbits up to period 7 ($\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\overline{0011}$, $\overline{0111}$, $\overline{00111}$, $\overline{001101}$, $\overline{001011}$, $\overline{0010111}$, $\overline{0011101}$, out of the 41 contributing to the noiseless, deterministic cycle expansion up to cycle period 7) suffice to fully determine the spectral determinant of the Fokker-Planck operator. In the evaluation of traces (13) we include stochastic corrections up to order $O(D)$ (an order beyond the term kept in (14)). The escape rate of the repeller of figure 1 so computed is reported in figure 3.

Since our 'optimal partition' algorithm is based on a sharp overlap criterion, small changes in noise strength D can lead to transition graphs of different topologies, and it is not clear how to assess the accuracy of our finite Fokker-Planck matrix approximations. We make three different attempts, and compute the escape rate for: (a) an under-resolved partition, (b) several deterministic, over-resolved partitions, and (c) a brute force numerical discretization of the Fokker-Planck operator.

(a) In the example at hand, the partition in terms of periodic points $\overline{00}$, $\overline{01}$, $\overline{11}$ and $\overline{10}$ is under-resolved; the corresponding escape rate is plotted in figure 3. (b) We

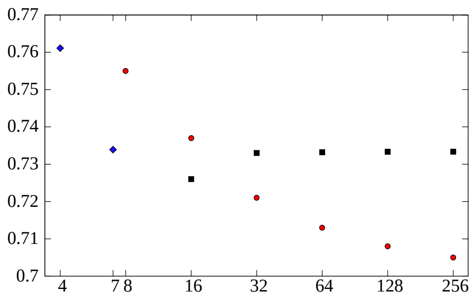

FIG. 3: The escape rate γ of the repeller figure 1 plotted as function of number of partition intervals N, estimated using: (♦) under-resolved 4-interval and the 7-interval 'optimal partition', (●) all periodic orbits of periods up to $n = 8$ in the deterministic, binary symbolic dynamics, with $N_i = 2^n$ periodic-point intervals (the deterministic, noiseless escape rate is $\gamma_{det} = 0.7011$), and (■) a uniform discretization (16) in $N = 16, \cdots, 256$ intervals. For $N = 512$ discretization yields $\gamma_{num} = 0.73335(4)$.

calculate the escape rate by over-resolved periodic orbit expansions, in terms of *all* deterministic periodic orbits of the map up to a given period, with t_p evaluated in terms of Fokker-Planck local traces (13), including stochastic corrections up to order $O(D)$. Figure 3 shows how the escape rate varies as we include all periodic orbits up to periods 2 through 8. Successive estimates of the escape rate appear to converge to a value different from the 'optimal partition' estimate. (c) Finally, we discretize the Fokker-Planck operator $\mathcal{L}$ by a piecewise-constant approximation on a uniform mesh on the unit interval [16],

$$[\mathcal{L}]_{ij} = \frac{1}{|\mathcal{M}_i|} \frac{1}{\sqrt{4\pi D}} \int_{\mathcal{M}_i} dx \int_{f^{-1}(\mathcal{M}_j)} dy \, e^{-\frac{1}{4D}(y-f(x))^2} , \quad (16)$$

where $\mathcal{M}_i$ is the ith interval in equipartition of the unit interval into N pieces. Empirically, $N = 128$ intervals suffice to compute the leading eigenvalue of the discretized $[128 \times 128]$ matrix $[\mathcal{L}]_{ij}$ to four significant digits. This escape rate, figure 3, is consistent with the $N = 7$ 'optimal partition' estimate to three significant digits.

In summary, we have presented a new method for partitioning the state space of a chaotic repeller in the presence of noise. The key idea is that the width of the linearized adjoint Fokker-Planck operator $\mathcal{L}_a^\dagger$ eigenfunction computed on a periodic point x_a provides the scale beyond which no further local refinement of state space is possible. This computation enables us to systematically determine the *optimal partition*, of the finest state space resolution attainable for a given chaotic dynamical system and a given noise. Once the optimal partition is determined, we use the associated transition graph to describe the stochastic dynamics by a *finite dimensional* Fokker-Planck matrix. While an expansion of the Fokker-Planck operator about periodic points was already introduced in ref. [13], the novel aspect of this work is its representation in terms of the eigenfunctions of the linearized Fokker-Planck operator (4), *ie.* the Hermite basis [10, 11].

This representation enables us to incorporate perturbatively the effects of weak noise in a novel way. We test our optimal partition hypothesis by applying it to evaluation of the escape rate of a $1d$ repeller in presence of additive noise. Numerical tests indicate that, as an example, a 7-interval 'optimal partition' can be as accurate as 128-interval numerical discretization of the Fokker-Planck operator. By contrast, cycle expansions based on deterministic symbolic dynamics (which assumes that arbitrarily fine resolution of the state space is meaningful) yield a value of the escape rate significantly different from the correct one.

The success of the optimal partition hypothesis in a 1-dimensional setting is encouraging. However, higher-dimensional hyperbolic maps and flows, for which an effective optimal partition algorithm would be very useful, present new challenges due to the subtle interactions between expanding, marginal and contracting directions. This will be addressed in a future publication [11].

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