PLANAR PERTURBATION EXPANSION

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Generating functionals for planar field theories are defined in terms of non-commuting sources. Relations between generating functionals for full, connected and one-particle-irreducible Green's functions are given, together with the Dyson–Schwinger equations which generate the planar perturbation expansion.

The planar diagrams constitute a very small fraction of all diagrams in ordinary field theories. Their number grows slowly, roughly like (constant)^n in nth order [1], while the number of all diagrams explodes factorially, like $n!$. Do there exist "planar field theories" whose perturbation expansions contain only planar diagrams? In this letter we formulate such theories. Our construction parallels the construction of fermion field theories; the key observation is that the algebra of sources should reflect the symmetry of the Green's functions. As planar diagrams have no symmetry under interchanges of external legs, the sources should be non-commuting. The structure of planar generating functionals that emerge from our construction is quite different from those of ordinary field theories. In particular, the exponentials of the ordinary theories are replaced by continued fractions in the planar theories.

Planar generating functionals. Let $Z_{ij\ldots k}$, $W_{ij\ldots k}$ and $\Gamma_{ij\ldots k}$ be the full, connected and one-particle-irreducible Green's functions, respectively. Here the index i stands for all variables needed to specify a particle, such as the momentum, spin, species, etc. Repeated indices imply summations over discrete labels and integrations over the continuous variables. If the symmetry of Green's functions is not specified, a generating functional can be constructed by introducing a source $J_i^{(1)}$ for the first particle, $J_i^{(2)}$ for the second particle, and so on:

$$Z[J^{(1)}, J^{(2)}, \cdots] = \sum_{m=0}^{\infty} Z_{ij\ldots k} J_k^{(m)} \ldots J_j^{(2)} J_i^{(1)} . \tag{1}$$

The Green's functions are retrieved by ordinary differentiation:

$$Z_{ij} = \frac{\mathrm{d}}{\mathrm{d}J_i^{(1)}} \frac{\mathrm{d}}{\mathrm{d}J_j^{(2)}} Z[J^{(1)}, J^{(2)}, \cdots]|_{J=0} . \tag{2}$$

However, functionals with labelled sources are useless if one wishes to study relations among different Green's functions. For Bose theories multiple sources are avoided by the observation that the Green's functions are fully summetric, and that this symmetry is respected by a singele c-number source J_i:

$$Z[J] = \sum_{m=0}^{\infty} Z_{ij\ldots k} \frac{J_k \cdots J_j J_i}{m!} . \tag{3}$$

The combinatoric factor ensures that the Green's functions are recovered by

$$Z_{ij\ldots k} = \frac{\mathrm{d}}{\mathrm{d}J_i} \frac{\mathrm{d}}{\mathrm{d}J_j} \cdots \frac{\mathrm{d}}{\mathrm{d}J_k} Z[J]|_{J=0} . \tag{4}$$

For fermions the Green's functions are fully antisymmetric, and their symmetry is respected by a single anticommuting source J_i : $\{J_i, J_j\} = 0$. Relation (4) applies, provided that the derivatives anticommute.

The planar Green's functions have no symmetry

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under interchanges of legs, as this in general gives rise to non-planar diagrams. However, also in this case the sources $J_i^{(1)}, J_i^{(2)}, \cdots$ can be replaced by a single source J_i, provided that the source is a non-c number (non-commuting), $J_iJ_j \neq \pm J_jJ_i$, $i \neq j$, and that the derivative is likewise non-commuting:

$$\frac{\mathrm{d}}{\mathrm{d}J_i}(J_jJ_k \cdots J_m) \equiv \delta_{ij}J_k \cdots J_m \,, \tag{5}$$

d/dJ is the operation of picking out the leftmost leg of a Green's function, and not the conventional c-number derivative. We use the derivative notation to emphasize the parallellism with the ordinary Bose and Fermi theories.

In terms of non-c sources the generating functionals for full and connected planar Green's functions are given by

$$Z[J] = 1 + \sum_{m=1} Z_{ij\ldots k} J_k \cdots J_j J_i \,,$$
$$W[J] = \sum_{m=1} W_{ij\ldots k} J_k \cdots J_j J_i \,. \tag{6}$$

and the Green's functions are again recovered by (4). (In the above sums m is the number of legs in the corresponding Green's function.)

To establish the relation between full and connected Green's functions, consider an arbitrary diagram contributing to a full Green's function. If we pull out the leftmost leg, we pick out a connected diagram whose remaining legs enter the diagram somewhere to the right, and interspersed between these legs are various disconnected bits (fig. 1a). Functionally this is expressed by a recursion relation

$$Z[J] = 1 + W[JZ[J]] \,. \tag{7a}$$

Had we started our construction by pulling out the rightmost leg, we would have obtained

$$Z[J] = 1 + W[Z[J]J] \,. \tag{7b}$$

Either relation immediately leads to the explicit solution of the free planar field theory. There is only one connected diagram

$$W[J] = \Delta_{ij}J_jJ_i \,, \tag{8}$$

and the generating functional for the full Green's functions is

$$Z[J] = 1 + \Delta_{ij}J_jZ[J]J_iZ[J]$$
$$= \{1 - \Delta_{ij}J_jZ[J]J_i\}^{-1} \,, \tag{9a}$$

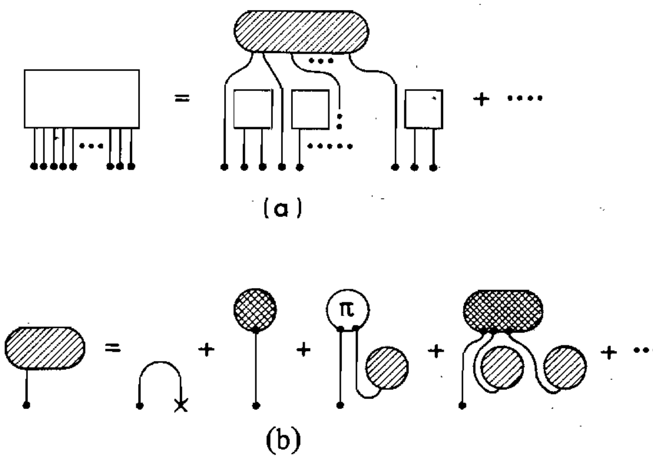

Fig. 1. (a) Relation between the full and the connected planar Green's functions. The full Green's functions are denoted by boxes, and the connected Green's function by hatched blobs. (b) Expansion of a connected Green's function in terms of one-particle-irreducible Green's functions. One-particle-irreducible Green's functions are denoted by crosshatched blobs. All sources J_i are implicitly contained in the connected Green's functions (hatched blobs).

or, iterating

$$Z[J] = \cfrac{1}{1 - J_i \cfrac{\Delta_{ji}}{1 - J_k \cfrac{\Delta_{lk}}{1 - J \cfrac{\Delta}{\vdots} J} J_l} J_j} \,. \tag{9b}$$

Here the fraction $1/(1-x)$ is shorthand for the expansion $1 + x + x^2 + \cdots$. Iteration of (9a) or expansion of (9b) now yields the free planar field theory (fig. 2).

Let us apply the reasoning that led to (7) to interacting planar field theories. Consider planar ϕ^3 theory, characterized by a bare vertex γ_{ijk}. Pulling out the leftmost leg we either end on some other external leg, or hit a vertex (fig. 3a). Repeating this procedure we obtain all planar diagrams up to a given order perturbation theory. In terms of $Z[J]$, this is the Dyson–Schwinger equation for the full planar Green's functions

$$\frac{\mathrm{d}}{\mathrm{d}J_i}Z[J] = \Delta_{ji}\left(Z[J]J_j + \gamma_{klj}\frac{\mathrm{d}}{\mathrm{d}J_l}\frac{\mathrm{d}}{\mathrm{d}J_k}\right)Z[J] \,.$$

The corresponding Dyson–Schwinger equation for connected Green's functions, fig. 3b, follows from (7). For

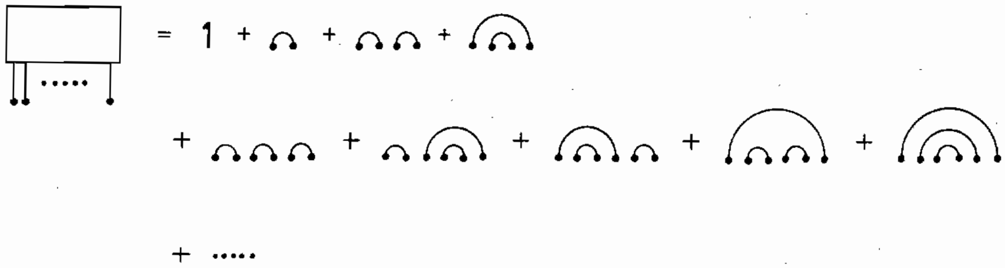

Fig. 2. The full Green's functions for the free planar theory.

an arbitrary theory, the bare vertices are given by the classical action

$$S[\phi] = -\Delta_{ij}^{-1}\phi_j\phi_i + \gamma_{ijk}\phi_k\phi_j\phi_i + \cdots , \qquad (10)$$

and (10) can be rewritten as the equation of motion for $Z[J]$:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}\phi_i}\left[\frac{\mathrm{d}}{\mathrm{d}J}\right] + Z[J]J_i\right)Z[J] = 0 . \qquad (11)$$

Next we turn to the construction of the effective action

$$\Gamma[\phi] = \sum_{m=1}^{\infty} \Gamma_{ij\ldots k}\phi_k\cdots\phi_j\phi_i . \qquad (12)$$

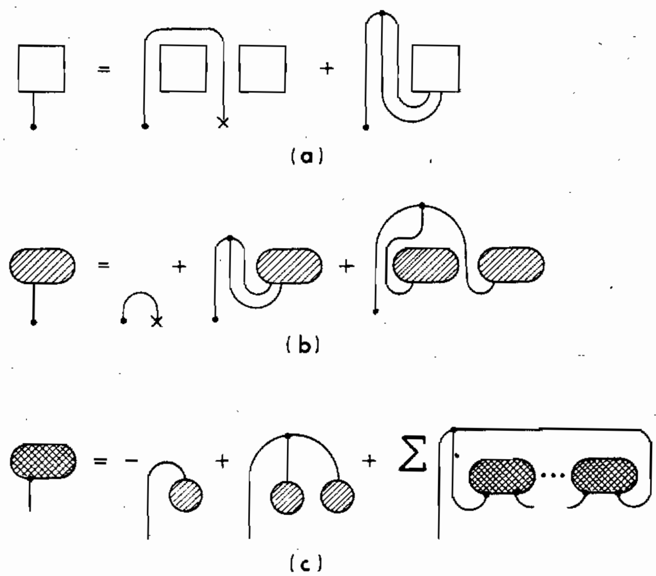

Fig. 3. Dyson–Schwinger equations for ϕ^3 theory. (a) Full Green's functions (sources J_i implicit) (b) Connected Green's functions (sources J_i implicit) (c) One-particle-irreducible Green's functions (fields ϕ_i implicit). As the sources and fields are non-c numbers, their ordering is crucial.

Take a connected planar diagram and pull out its leftmost leg. One possibility is that this leg immediately leaves the diagram. Otherwise it ends in a proper tadpole, or a proper self-energy which continues into a connected piece, or in a proper vertex which then continues into two connected pieces, etc. (fig. 1b). Functionally this is expressed by

$$\phi_i \equiv \mathrm{d}W[J]/\mathrm{d}J_i , \qquad (13)$$

$$\phi_l = \Delta_{il}(J_i + \Gamma_i + \pi_{ji}\phi_j + \Gamma_{kji}\phi_j\phi_k + \cdots)$$
$$= \Delta_{il}(J_i + \mathrm{d}\Gamma_\mathrm{I}[\phi]/\mathrm{d}\phi_i) , \qquad (14)$$

where the "interaction" functional $\Gamma_\mathrm{I}[\phi]$ is related to (12) by

$$\Gamma[\phi] = \Gamma_\mathrm{I}[\phi] - \Delta_{ij}^{-1}\phi_j\phi_i .$$

Rewriting (14) in terms of Γ we obtain the equation of motion for the effective planar action:

$$\mathrm{d}\Gamma[\phi]/\mathrm{d}\phi_i + J_i = 0 . \qquad (15)$$

The Dyson–Schwinger equations (fig.3c) now follow from fig. 3b. (13) and (15) now also give us the Legendre transform for planar generating functionals:

$$W[J] = \Gamma[\phi] + \phi_i J_i + J_i\phi_i . \qquad (16)$$

This completes the generating functional formalism for planar field theories. The key results are equation (7), which says that planar field theory is characterized by continued fractions (rather than exponentials) and equation (16) which gives the Lagrange transformation for non-commuting sources and fields.

Planar path integrals. From experience with the fermionic functionals we know that path integrals are not necessarily integrals; rather, they are rules which map the operation d/dJ into a field ϕ. Guided by the observation that the classical action S is the tree approximation to the effective action Γ, we guess that the planar path integral is of form

$$Z[J] = \int [\mathrm{d}\phi]\, \widetilde{Z}[\phi, J] ,$$
$$\widetilde{Z}[\phi, J] \equiv \widetilde{Z} = 1 + S[\phi\widetilde{Z}] + \{\phi_i\widetilde{Z}, J_i\widetilde{Z}\} . \quad (17)$$

Here the continued fraction structure is motivated by (7) (which is the planar theory's equivalent of exponentiation in the full theory), and the anticommutator by the Legendre transform (16). The fields ϕ_i are non-c numbers. $\int[\mathrm{d}\phi]$ is defined operationally; we require that the order of the $\mathrm{d}/\mathrm{d}J$, $\int[\mathrm{d}\phi]$ operations can be interchanged. This relates Green's functions to the expectation values of products of fields:

$$\langle \phi_i\phi_j \cdots \phi_k \rangle \equiv \frac{\mathrm{d}}{\mathrm{d}J_i}\frac{\mathrm{d}}{\mathrm{d}J_j} \cdots \frac{\mathrm{d}}{\mathrm{d}J_k} Z[J]$$
$$= \int [\mathrm{d}\phi]\, \widetilde{Z}\phi_i \widetilde{Z}\phi_j \cdots \widetilde{Z}\phi_k \widetilde{Z} . \quad (18)$$

Hence $\mathrm{d}/\mathrm{d}J_i \leftrightarrow \phi_i$, as desired.

The other requirement we impose on $\int[\mathrm{d}\phi]$ is that it should respect translational invariance, which we state as the absence of surface terms in integration over total derivatives: $\int[\mathrm{d}\phi]\, \mathrm{d}(\cdots)/\mathrm{d}\phi_i = 0$. Applying this to $\widetilde{Z}[\phi, J]$ we obtain

$$0 = \int [\mathrm{d}\phi]\, \frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi_i} = \int [\mathrm{d}\phi] \left(\frac{\mathrm{d}S}{\mathrm{d}\phi_i}[\widetilde{Z}\phi] + \widetilde{Z}J_i \right) \widetilde{Z} , \quad (19)$$

i.e., the Dyson–Schwinger equation (11). We emphasize that the above path integral is just a guess. All information is contained in the generating functional (6). The reader might wonder why we have not mentioned cyclic symmetry yet? That is because planar Green's functions need not be cyclic; they are cyclic if the starting complete theory had Bose symmetry. In terms of our derivatives (5), cyclicity means that identities like

$$(\mathrm{d}/\mathrm{d}\phi_i) S[\phi]\, \phi_i = S[\phi] , \quad (20)$$

are satisfied.

Zero-dimensional theories. No solution of planar Dyson–Schwinger equations is known except for the zero-dimensional planar field theories. These are diagram counting expansions in terms of a single source $J_i = J$, so that the generating functionals commute and the differentiation (5) is simply

$$\mathrm{d}Z[J]/\mathrm{d}J = \{Z[J] - 1\}/J .$$

The Dyson–Schwinger equations are now ordinary polynomial equations. These equations are solved for a number of theories in ref. [1]. The most trivial example is the generating functional for the free field theory (9), which becomes

$$Z[J] = \frac{1 - (1 - 4J^2)^{1/2}}{2J^2} \sum_{n=0} \frac{(2n)!}{n!(n+1)!} J^{2n} \quad (21)$$
$$= 1 + J^2 + 2J^4 + 5J^6 + \cdots$$

Here the coefficient of J^{2n} is the number of diagrams with $2n$ legs (fig. 2).

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