Abstract—High dimensional spatiotemporal chaos underpins the dynamics associated with life threatening cardiac arrhythmias. Our research seeks to suppress arrhythmias and restore normal heart rhythms by using low-energy transfers between recurrent unstable solutions embedded within the chaotic dynamics of cardiac tissue. We describe the search for these unstable solutions in nonlinear reaction diffusion models of cardiac tissue and discuss recent efforts to reduce local symmetries that emerge as a result of spiral wave interactions.

I. INTRODUCTION

Sudden cardiac arrest is the leading cause of death in the industrialized world. The majority of these deaths are due to cardiac arrhythmic disorders such as ventricular fibrillations caused by myocardial infarctions, or “heart attacks.”

Arrhythmias are characterized by spatially complex, high-dimensional dynamics that occur as excitation waves propagate through cardiac tissue. Figure 1 shows an example of how ventricular tachycardia (an arrhythmic behavior characterized in two dimensions by spiral wave dynamics) can deteriorate into ventricular fibrillation—a dangerous chaotic regime characterized by a loss of spatial coherence in the contractile dynamics of the heart. While in this state, the heart’s ability to pump blood is severely degraded, and death usually follows within minutes unless corrective measures are taken.

The most common and effective way to terminate ventricular fibrillation is by using a defibrillator—an external or implanted medical device that delivers one or more high energy electrical shocks to forcefully “reset” the excitation dynamics of cardiac tissue. This “brute force” methodology reestablishes the normal sinus rhythm of the heart, but causes a variety of undesired side effects including severe pain, cardiac tissue damage, and a rapid loss of battery life in implanted devices [1], [2]. Our research seeks to improve the defibrillation process by using a novel dynamical systems approach that provides a safer, lower energy alternative with fewer negative side effects.

The dynamical systems approach to low-energy control is based on two equivalent representations of the cardiac dynamics: (1) a physical space representation and (2) a state space representation. In the former, the state of the tissue is described by a vector field \( \mathbf{w} \) where the vector \( \mathbf{w}(x,y,t) \) describes the state at time \( t \) of a cardiac cell (cardiomyocyte) located at position \( (x,y) \). The latter uses an arbitrary alternative representation of the vector field \( \mathbf{w} \) that is convenient for computational purposes, e.g., the values of \( \mathbf{w} \) on a computational mesh or Fourier coefficients of \( \mathbf{w} \) in the spectral representation.

The advantage of the state space representation is that it allows solutions in the physical space to be described in terms of steady-, time-periodic and quasi-periodic solutions that are known to organize chaotic dynamics in state space. The physical space versions of these solutions are conventionally referred to as exact coherent states due to their similarity with the spatiotemporal structures (coherent states) frequently observed in turbulent fluid flow experiments.

These solutions and their dynamical (e.g., heteroclinic) connections provide ideal pathways to guide the system away from the dangerous regions of state space associated with cardiac arrhythmias and back towards safe regions where normal rhythms can be restored. The theory based on these unstable solutions has already enabled impressive progress in understanding space manifold dynamics and other high dimensional chaotic systems such as weakly turbulent fluid flows [3], [4], [5], [6].

Numerical methods originally developed in the context of fluid turbulence have enabled us to identify several invariant solutions in nonlinear reaction-diffusion models of cardiac tissue. These include equilibria and relative equilibria—traveling wave solutions (e.g. pulse and spiral waves) which respect global translational and rotational symmetries in unbounded domains [7], [8] but which become time-periodic solutions in bounded domains.

We have also identified numerous multi-spiral solutions which are not strictly time-periodic because of local symmetries that form in the presence of weakly interacting spiral waves. In order to find exact periodic solutions featuring many interacting spirals, a new approach is needed that incorporates the dynamics associated with local rather than global symmetries. The reduction of these local symmetries

Fig. 1: An unstable spiral wave formed during ventricular tachycardia breaks up into multiple interacting spirals. This process leads to the chaotic activity observed during ventricular fibrillation. Simulations are performed using the Karma model defined by Eqs. (1-8) with “no flux” boundary conditions.
will provide an important step towards obtaining the sets of unstable periodic solutions needed to implement an optimal low-energy control.

This work is organized as follows. In Sec. II we describe the computational procedure for locating unstable recurrent solutions and show examples of equilibria, relative equilibria and some nearly time-periodic trajectories. In Sec. III we illustrate the effects of local symmetry by constructing multi-spiral solutions from a single-spiral solution. In Sec. IV we introduce the concept of tiling to partition the domain for local symmetry reduction. In Sec. V we provide some conclusions.

II. THE SEARCH FOR COHERENT STRUCTURES

Many modern mathematical models of cardiac tissue are algebraically complex and incorporate tens of variables per cell. The tissue dynamics are typically described in terms of coupled nonlinear reaction-diffusion partial differential equations

\[ \partial_t w = D \nabla^2 w + f(w), \]  

where the vector field \( w = (u, v) \) represents the transmembrane voltage and gating variables (ionic concentrations, state of ionic channels, etc.) for the cardiac muscle cells making up the tissue, \( D \) is a diagonal matrix of diffusion coefficients, and the nonlinear function \( f(w) \) describes the local dynamics of individual cardiac cells.

These models require substantial computational resources to obtain numerical solutions with appropriate spatial and temporal resolution. When compared to some algebraically simpler, lower dimensional models, the added complexity provides little additional insight into the dynamical mechanisms responsible for generating and maintaining complex arrhythmic behaviors.

The two-variable Karma model [9] provides a greatly simplified description of cardiomyocyte excitation dynamics while still reproducing the essential alternans instability that is responsible for the breakup of single-spiral solutions and the transition to fibrillation in two or three dimensions. The cellular dynamics of this model are described by the following function

\[ f(w) = \left( \frac{u^2(1 - \tanh(u - 3))(u^* - v^*)/2 - u}{\epsilon((\beta^{-1} - v)\Theta(u - 1) - \Theta(1 - u)v)} \right), \]  

where \( \Theta(\cdot) \) is the Heaviside step function and \( (\epsilon, u^*, \beta) \) are the control parameter set. The parameter \( \epsilon \) describes the ratio of excitation and relaxation time-scales, \( u^* \) is a phenomenologically chosen voltage scale, and \( \beta = 1 - \exp(-Re) \) controls the restitution (and susceptibility of traveling excitation waves to alternans, or period-doubling instability) through a secondary control parameter \( Re \). We chose the parameters \( u^* = 1.5415, \epsilon = 0.01 \) and \( Re = 1.273 \) such that simple traveling wave solutions such as those shown in Fig. 2(a-b) would break up due to alternans instability.

The equations are solved on an \( N \times N \) uniform grid with spacing \( \Delta = 262 \mu m \) which corresponds to the size of cardiomyocytes. No-flux boundary conditions are used unless otherwise specified. At each time instant \( t \), the discretized fields \( u \) and \( v \) that define the state of the system can be thought of as an element \( z(t) \) of a \( 2N^2 \)-dimensional vector space \( \mathcal{M} \subset \mathbb{R}^{2N^2} \). Under time evolution, \( z(t) \) defines a trajectory in this high-dimensional state space.

The first class of recurrent solutions are the equilibria. In the absence of spatial variation, the Laplacian term vanishes and the reaction diffusion equations are reduced to a set of \( 2N^2 \) identical uncoupled ordinary differential equations whose solutions represent the dynamics of individual cells. Equilibrium points for the ordinary differential equations can be obtained by finding the intersections of the nullclines defined by \( f(w) = 0 \). Three such solutions exist for the choice of parameters given above: a stable (but excitable) equilibrium point at the origin \( w^*_1 = (0, 0) \), a saddle point at \( w^*_2 = (0.6547, 0) \) and an unstable focus at \( w^*_3 = (1.0073, 0.853) \). Each of these equilibrium points also provides a steady state solution for the tissue model when \( D \neq 0 \).

Another class of unstable solutions exist due to the Euclidean symmetry of the underlying equations. These solutions are known as relative equilibria and describe traveling waves. Two examples are shown in Fig. 2(a) and (b). For the plane traveling wave shown in Fig. 2(a), time evolution is equivalent to a global translation in the horizontal direction. For the spiral traveling wave shown in Fig. 2(b), time evolution is equivalent to a global rotation about a zero-dimensional core [10] that pins the spiral.
reduced representation, where continuous translational and rotational symmetries are factored out of the dynamics, both types of relative equilibria reduced to an equilibrium [7].

A slightly more complex type of exact coherent structures are described by unstable time-periodic solutions. These solutions can be found using the method of close returns. This method is based on the observation that the difference $E(t, \tau) \equiv \|w(t) - w(t - T)\|$ is small when a state space trajectory $w(t)$ shadows an unstable periodic orbit of period $T$, where $\| \cdot \|$ denotes the $L^2$ norm in $\mathbb{R}^{2N^2}$. In contrast, $| \cdot |$ will be used below to denote the absolute value.

The recurrence plot shown in Fig. 3 illustrates how we identify nearly recurrent solutions. Once a sufficiently low minimum of the recurrence function $E(t, \tau)$ is identified in the plot, a matrix-free Newton-Krylov solver (GMRES) is used to refine the nearly time-periodic solution into an exact time-periodic solution. Computational challenges associated with the high dimensionality of the state space $\mathcal{M}$ are handled by developing heavily parallelized numerical integrators [11] and Krylov-space algorithms that use powerful NVIDIA GPU hardware.

Nearly exact periodic solutions are produced when the Newton-Krylov solver fails to converge. Figure 2(c) shows an example of such a nearly recurrent, multi-spiral solution with a period $T = 377.95 \text{ ms}$ and a relative residual of $\|w(T) - w(0)\|/\|w(0)\| = 5 \times 10^{-4}$. The convergence failure can be understood by examining the residual distribution in the spatial domain, shown in Fig. 2(d). The residual is concentrated near the wave fronts and cores of three specific spiral waves that undergo slow drifts. These drifts are associated with local symmetries.

III. LOCAL SYMMETRIES

Spiral waves interact very weakly when separated by large distances. This means that neither their relative positions, nor their relative phases are fixed. The ability to apply small displacements to a core or small shifts to its phase (which is equivalent to a spatial rotation of the spiral about its core) relative to the other cores implies that the solutions respect Euclidean symmetry both globally and locally.

We illustrate the local Euclidean symmetries encountered in the dynamics by constructing two simple multi-spiral solutions. The first solution is constructed by placing a copy of the single, isolated spiral solution shown in Fig. 2(b) next to a second copy that has been integrated forward in time by a quarter of its temporal period (which is equivalent to a spatial rotation of this spiral by $\pi/2$). The resulting solution is shown in Fig. 4.

The cartoon description of the spiral core motion shown in the middle of Fig. 4 shows that there is no net drift. Unlike the single-spiral state (which is properly classified as a relative equilibrium), the two-spiral state is a time-periodic solution. The two-spiral solution recurs exactly (within the numerical precision of the simulation) after a period $T = 125.98 \text{ ms}$. In this case the Newton-Krylov solver (which automatically reduces global symmetries) succeeds in finding the two-spiral solution because each of the spirals is characterized by the same type of local symmetry (a phase shift/rotation).

The stability of a recurrent solution in state space is determined from the eigenvalues (stability multipliers) $\lambda_k$ and corresponding eigenvectors of the finite-time jacobian $J^\text{fp}_k(z_p) = \partial z(T_p)/\partial z(t_0)$, where $J^0(z_p) = I$, $z_p$ is the initial condition on the state space orbit and $T_p$ is the time to complete the orbit. Because of the high dimensionality of the state space, the leading stability multipliers are determined numerically using Arnoldi iteration. The stability exponents $\lambda_k = \mu_k + i\omega_k$ are related to the stability multipliers $\lambda_k = e^{T_p \mu_k}$ whose magnitude $|\lambda_k| = e^{T_p \mu_k}$ determines whether small perturbations grow or decay under the dynamics.

The multipliers for the phase shifted, two-spiral solution are plotted in Fig. 5 along with some of the corresponding eigenmodes for the unstable (expanding) $|\lambda| > 1$, stable (contracting) $|\lambda| < 1$ and marginal $|\lambda| = 1$ directions. The unstable modes in the spectrum are associated with an alternans instability which is responsible for the spiral breakup shown in Fig. 1. The pair of marginal modes found in the spectrum indicate that either spiral can undergo small phase shifts (or rotations) relative to the other without destroying the time-periodic nature of the solution on the
Fig. 5: The leading stability multipliers for the phase shifted, two spiral exact coherent structure shown in Fig. 4. Several eigenmodes corresponding to the unstable and marginal directions are also shown. The unstable modes are associated with an alternan instability while the two marginal modes indicate that the solution contains a local symmetry in which either spiral is free to undergo small phase shifts relative to the other.

entire domain.

A nearly periodic, three-spiral solution with a residual of $3.5 \times 10^{-5}$ is shown in the top of Fig. 6. This solution illustrates a local symmetry involving core drift. The bottom of Fig. 6 shows that the residual is localized to the vicinity of the smaller drifting core. The failure of the Newton-Krylov solver to converge to a time-periodic three-spiral solution in this case is likely due to the difference in the types of the local symmetry for each of the three spirals—rotation for the two large spirals vs. a combined rotation and spatial translation for the small spiral. This failure requires development of a new approach that identifies regions in the domain where symmetry reduction can be applied locally.

IV. LOCAL SYMMETRY REDUCTION

Spiral waves naturally divide the domain by forming thin walls known as shock lines along which the phases of neighboring spirals are equal. The resulting sub-domains, or tiles, are ideal for local symmetry reduction because they each enclose a single-spiral.

Tiles were constructed in [12] for the complex Ginzburg-Landau equation (CGLE) by using the local amplitude of oscillation. Unlike the CGLE which describes weak nonlinear oscillations, the Karma model is strongly nonlinear, making it difficult to define, let alone compute the local amplitude. Instead, we use cycle areas in the $(u,v)$ plane to define a measure equivalent to the oscillation amplitude.

Figure 7(a) plots the instantaneous $(u,v)$ values at each cell on the computational grid for the three-spiral solution shown in Fig. 6. Over time, the dynamics of each oscillator trace out a continuous cycle (with a corresponding local period) whose area can be computed after each full revolution. Each point in the figure is colored by the area of its cycle. We find that the larger cycles (red) correspond to the shock lines between spirals, while the smaller cycles (blue) identify the

Fig. 6: A three-spiral solution with local symmetry in which the small core in the center drifts relative to the two larger spirals. The voltage (top), core trajectories (middle), and the residual (bottom) are shown. The $L^2$-norm of the residual is small but non-vanishing and is concentrated near the slowly drifting spiral core in the center of the domain.

Fig. 7: The oscillatory dynamics for each grid point are used to construct tiles on the domain. (a) The three-spiral solution from Fig. 6 shown in the $u-v$ representation. Each point is colored by the area its cycle traces out during a full revolution. The cycle area is useful in determining the location of phase singularities (spiral cores) and shock lines of uniform phase. (b) Three important cycles are extracted from (a). Ordered from the smallest to largest area, they correspond to points at a spiral core (blue), in the interior of a tile (lighter red), and on a tile boundary (dark red).
phase singularities that represent the spiral cores. The tiles defined by the cycle areas for the two-, three-, and ten-spiral solutions from Figs. 4(a), 4(b) and 2(c) are shown in Fig. 8.

Similar representations have been found useful for describing frozen/glassy spiral states in the CGLE [12], [13]. The authors show that tiles do not correspond to Voronoi tessellation. Instead, the shocks are approximated as segments of hyperbolae with the two nearest spiral cores as foci. Future work will determine whether, and under what conditions, the shock lines for the solutions of the Karma model are hyperbolic.

The decomposition of the domain into a set of weakly interacting tiles greatly simplifies the dynamical notion of invariance. On short time-scales, the pattern and temporal evolution within each tile is defined by a single-spiral solution. Because the interactions with neighboring spirals are weak, they can, to leading order, be neglected. This allows each tile to be treated as its own independent domain where the evolution of the spiral wave is affected by the action of a “global” symmetry. The independence of the tiles allows us to symmetry reduce the local (on the tile) solutions before recombining them to find global (on the entire domain) solutions that organize spatiotemporally chaotic dynamics.

Symmetries and tile geometries on the domain have important consequences for the globally recombined solutions. On a tile with an unbounded or circular domain, a spiral solution corresponds to a relative equilibrium. On a generic tile of finite size, a spiral solution will correspond to a time-periodic state since the geometry will typically breaks the continuous rotational symmetry. For stationary tiles (i.e., those that contain spirals with pinned cores), a time-periodic solution in the entire domain would naturally decompose into time-periodic solutions within each of the tiles. For drifting tiles (i.e., those whose boundaries deform because of drifting cores), the corresponding spirals would be described by relative periodic solutions. Hence, more generally, a class of multi-spiral solutions in the entire domain can be decomposed into periodic and relative time-periodic solutions inside the tiles.

V. Conclusions

The ability to understand and reduce local symmetries is essential for classifying and computing unstable time-dependent solutions of cardiac tissue models. These solutions form a skeleton for the spatiotemporally chaotic dynamics associated with cardiac arrhythmias in their state space representation. A dynamical description of arrhythmic behaviors based on transitions between such solutions provides an important first step towards designing efficient low-energy methods to suppress fibrillation. We have developed a computational procedure for finding time-periodic solutions in the state space representation corresponding to single- and multi-spiral states with fixed spiral cores in the physical space representation.

In its current form, this procedure is unable to find solutions in which spiral waves drift relative to one another. As a result, local symmetries continue to mask many of the unstable recurrent solutions that are important for implementing an effective method of low-energy control. The ability to partition the spatial domain into tiles on which the dynamics effectively decouple on short time-scales should provide a framework for a systematic application of local symmetry reduction both in models of cardiac tissue and other systems displaying spatiotemporally chaotic dynamics.

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