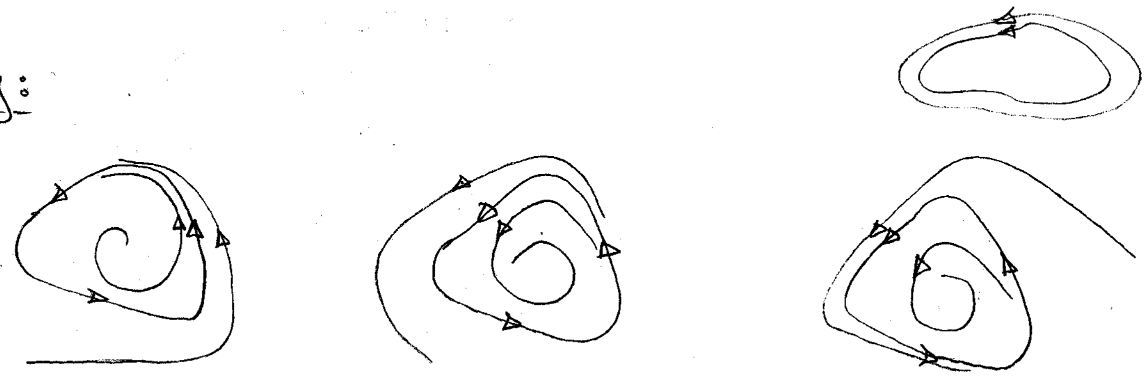
Limit Cycles

Definition: limit cycle is an isolated closed trajectory

Consequence: Linear systems cannot have limit cycles:
if $\vec{x}(t)$ is a closed orbit, $c\cdot\vec{x}(t)$ is too.

Stability:

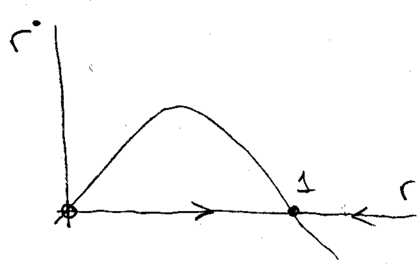


stable unstable half-stable

Example:

$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$$

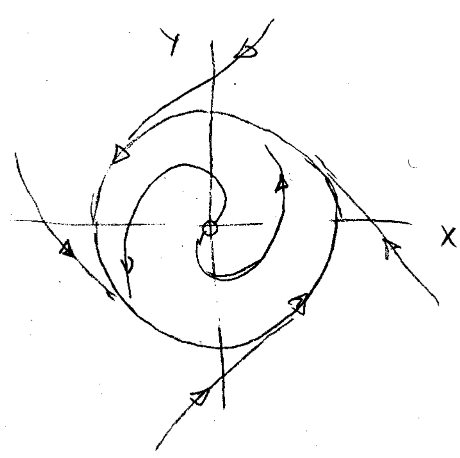
$\Rightarrow r^* = 1$ - fixed point for 1-d system



$r(t) \to r^* = 1$, $t\to\infty$ (stable)

$\theta(t) = \theta_0 + t$

$$\begin{cases} x(t) = r(t)\cos(\theta_0+t) \\ y(t) = r(t)\sin(\theta_0+t) \end{cases}$$



Existence of Closed Orbits

Existence or non-existence can be proved for certain classes of systems:

① Gradient Systems:

$$\dot{\vec{x}} = -\vec{\nabla} V(\vec{x}) \quad , \quad V \in C^1 \text{ - "potential"}$$

Theorem: Closed orbits are impossible for gradient systems

$$0 = \Delta V = \oint_C dV = \int_0^T \frac{dV}{dt} dt = \int_0^T -\left(\vec{\nabla} V \cdot \frac{d\vec{x}}{dt}\right) dt = -\int_0^T |\dot{\vec{x}}|^2 dt \leq 0$$

($\Delta V = 0$, if $|\dot{\vec{x}}|^2 \equiv 0$, i.e. $\vec{x}(t) \equiv \vec{x}^*$ - fixed point)

Example:

$$\begin{cases} \dot{x} = \sin y = -\frac{\partial}{\partial x}(-x \sin y) \\ \dot{y} = x \cos y = -\frac{\partial}{\partial y}(-x \sin y) \end{cases} \Rightarrow V(x,y) = -x \sin y$$

Necessary condition: $\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} \Rightarrow \frac{\partial}{\partial x} \dot{y} = \frac{\partial}{\partial y} \dot{x}$

Above: $\frac{\partial}{\partial x}(x \cos y) = \cos y = \frac{\partial}{\partial y}(\sin y)$ ✓

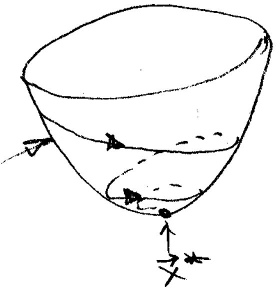
② Lyapunov Functions (in higher dimensions)

Definition: $V(\vec{x})$ - Lyapunov function, if $\exists \vec{x}^*$ - fixed point, and

1) $V(\vec{x}) > 0$, $\forall \vec{x} \neq \vec{x}^*$, and $V(\vec{x}^*) = 0$

2) $\frac{dV}{dt} < 0$, $\forall \vec{x} \neq \vec{x}^*$

No closed orbits: trajectories run "downhill"



Liénard theorem: Suppose

1) $f, g \in C^1$
2) $g(-x) = -g(x)$ — odd function
3) $g(x) > 0, \quad x > 0$
4) $f(-x) = f(x)$ — even function
5) $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero $x = a$,
   $F(x) < 0, \quad 0 < x < a$
   $F(x) > 0$, nondecreasing, $x > a$
   $F(x) \to \infty, \quad x \to \infty$

then Liénard eqn. has a unique, stable limit cycle surrounding the fixed point $x = \dot{x} = 0$

Example (van der Pol oscillator)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$$f(x) = \mu(x^2 - 1) \Rightarrow F(x) = \mu\left(\frac{x^3}{3} - x\right) \Rightarrow a = \sqrt{3}$$

$$g(x) = x$$

• x-small: $\ddot{x} - \mu\dot{x} + x = 0$

$$x \sim e^{\lambda t}: \quad \lambda^2 - \mu\lambda + 1 = 0 \Rightarrow \lambda = \begin{cases} \frac{\mu + \sqrt{\mu^2 - 4}}{2} \\ \frac{\mu - \sqrt{\mu^2 - 4}}{2} \end{cases} \to Re\,\lambda > 0$$

$\mu < 2$ — unstable spiral
$\mu > 2$ — unstable node

x-large: $\ddot{x} + \underbrace{\mu x^2}\dot{x} + x = 0$

large damping for large $|x|$