Bifurcations in 2-d

Recall: Bifurcation is a change in topology of the phase portrait (discontinuous) under smooth (continuous) change of parameters

Solutions:
- fixed points
- limit cycles

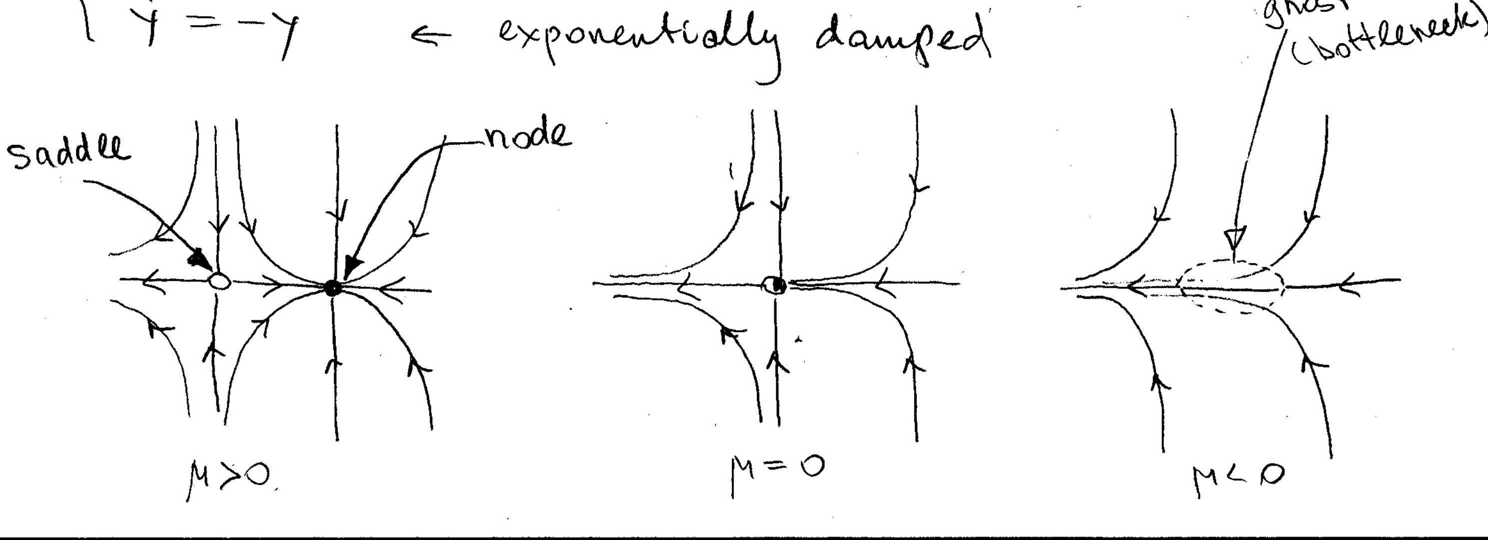
solutions are
- created
- destroyed
- change stability

Bifurcations of Fixed Points

Qualitatively similar to 1-d:

Example: (Saddle-Node)

$$\begin{cases} \dot{x} = \mu - x^2 & \leftarrow \text{1-d bifurcation at } \mu = 0 \\ \dot{y} = -y & \leftarrow \text{exponentially damped} \end{cases}$$



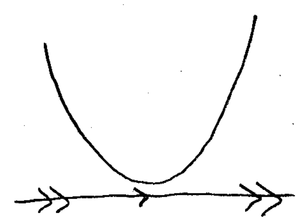
Comments: a) Idealized situation:

- straight (un)stable manifold
- Manifolds at right angle

(Generically: curved manifolds at arbitrary orientation)

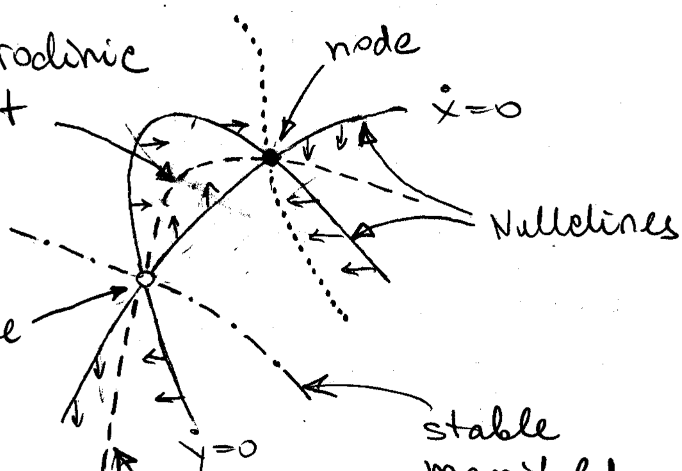
b) Ghost: when f.p. are destroyed, trajectories go through the "bottleneck" in time

$$T \approx \int_{-\infty}^{\infty} \frac{dx}{\mu + x^2} = \frac{\pi}{\sqrt{\mu}}$$

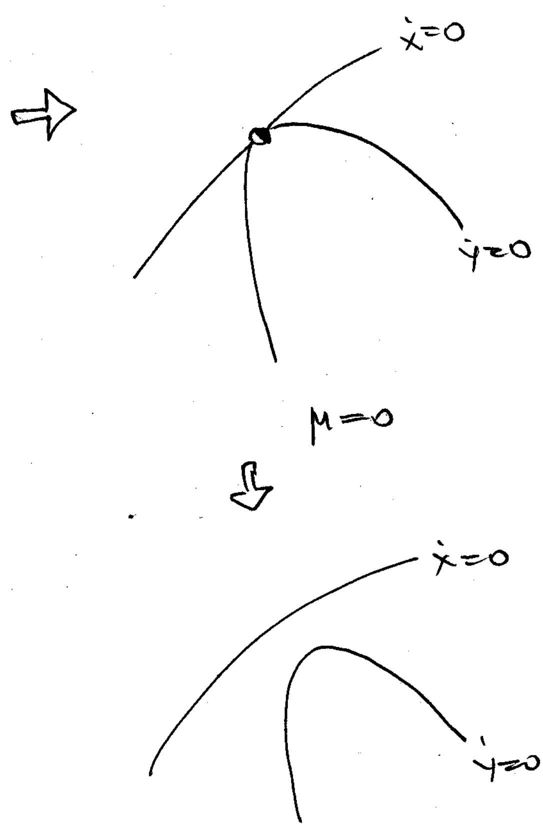


Nullclines:

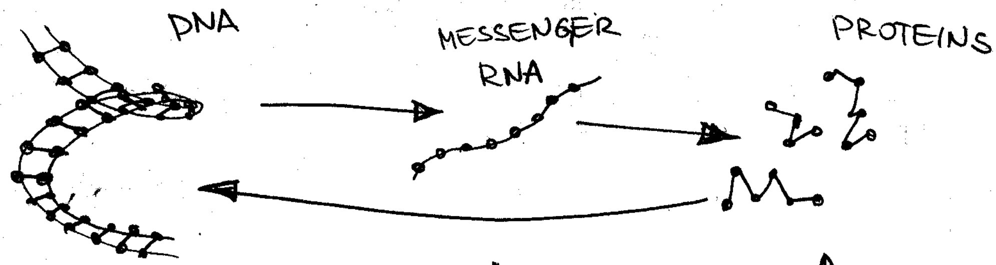
curves on which $\dot{x}=0$ or $\dot{y}=0$:



$\mu > 0$



Example (Genetic Control System)



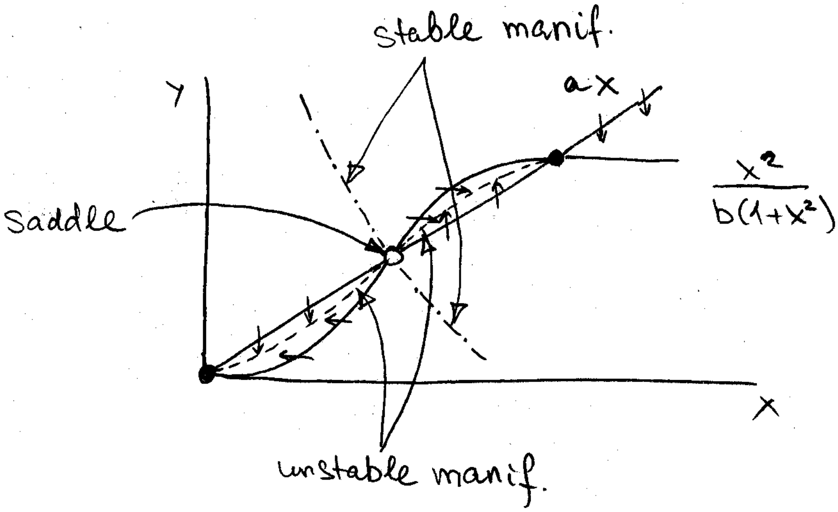
Concentrations: y x

Autocatalitic feedback:

$$\begin{cases} \dot{x} = y - ax \\ \dot{y} = \dfrac{x^2}{1+x^2} - by \end{cases}$$

degradation rates

Nullclines:



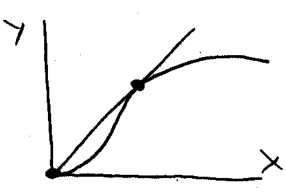
Fixed points:

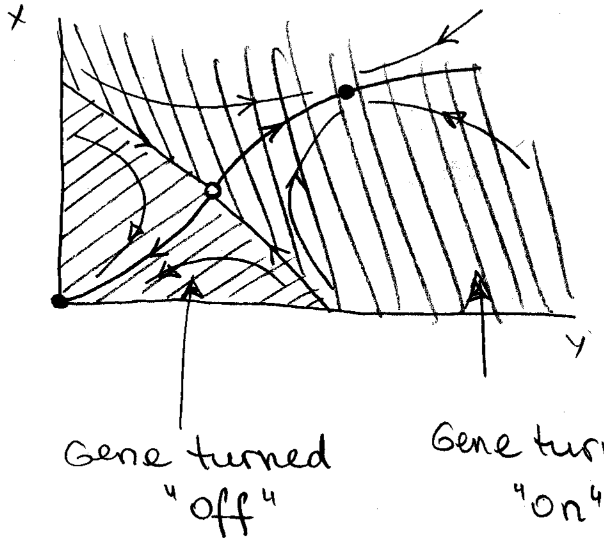
a) $x^* = y^* = 0$ – always f.p.

b) two others: nullclines intersect at $ax = \dfrac{x^2}{b(1+x^2)}$

$$\Rightarrow \quad ab\,x^2 - x + ab = 0 \quad \Rightarrow \quad x^*_{1,2} = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$$

Coalesce: $2ab = 1 \Rightarrow$ saddle-node bifurcation at $x^* = 1$





← Basins of attraction for $ab < \frac{1}{2}$ weak degradation

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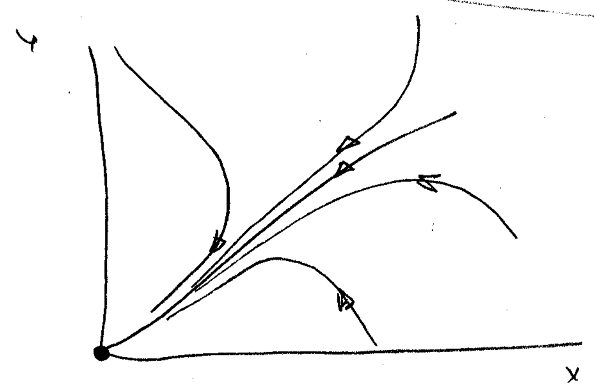
Single Basin for $ab > \frac{1}{2}$ →



Independent of initial conditions!

Example:

$$\begin{cases} \dot{x} = \mu x + y + \sin x \\ \dot{y} = x - y \end{cases}$$

← Invariant w.r.t. $(x,y) \to (-x,-y)$
⇒ Pitchfork bifurcation

Fixed point: $x^* = y^* = 0$ – always

Jacobian: $A = \begin{pmatrix} \mu+1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \Delta = -\mu - 2,\ \tau = \mu$

$$\Rightarrow \begin{cases} \text{saddle}, & \mu > -2 \\ \text{stable node}, & \mu < -2 \end{cases}$$

Sub- or Super-critical?

Other fixed points require $\dot{y} = x - y = 0 \Rightarrow x^* = y^*$

$$\Rightarrow \dot{x} = \mu x + x + \sin x = (\mu+1)x + x - \frac{x^3}{3!} + O(x^5) = 0$$

$$\Rightarrow \dot{x} = (\mu+2)x - \frac{1}{6}x^3 \approx 0 \quad \Rightarrow x^* \approx \pm\sqrt{6(\mu+2)}$$

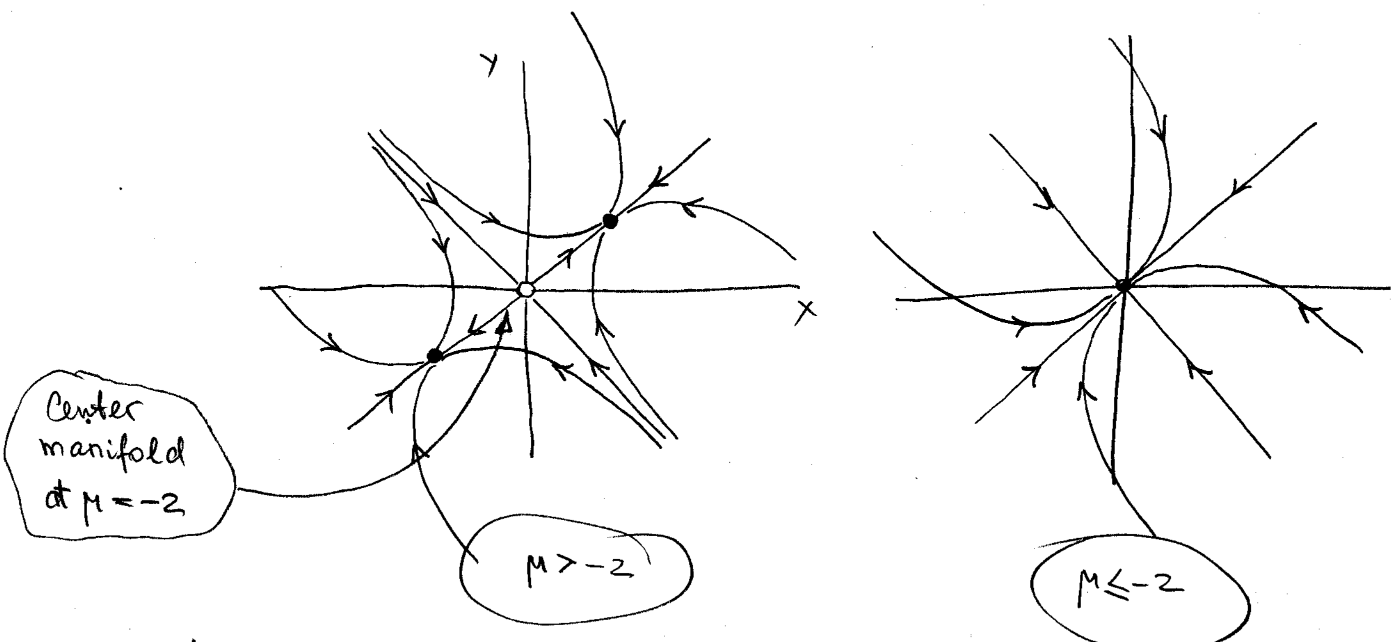
Get real solutions for $\mu > -2 \Rightarrow$

Supercritical pitchfork bifurcation at $\mu_c = -2$

Phase portrait:

Near bifurcation point $\mu \approx -2$, $A \approx \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

Eigenvectors: $\vec{e}_1 \approx (1,1)$, $\vec{e}_2 \approx (1,-1)$;
Eigenvalues: $\lambda_1 \approx 0$, $\lambda_2 \approx -2$



Saddle-Node, Pitchfork (also transcritical):

One eigenvalue crosses 0: $\lambda_i \sim (\mu - \mu_c)$

two eigenvalues → Hopf bifurcation