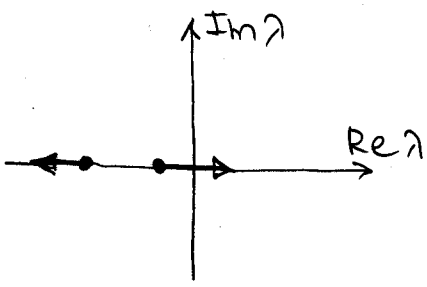
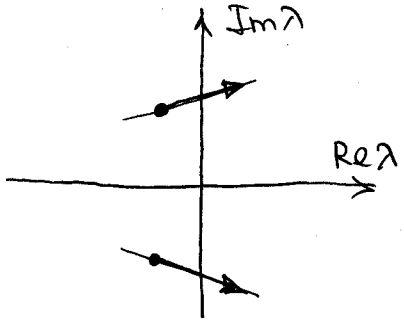
Hopf Bifurcation

- So far looked at f.p. ↔ f.p.
- Now : f.p. ↔ limit cycle

Eigenvalues:

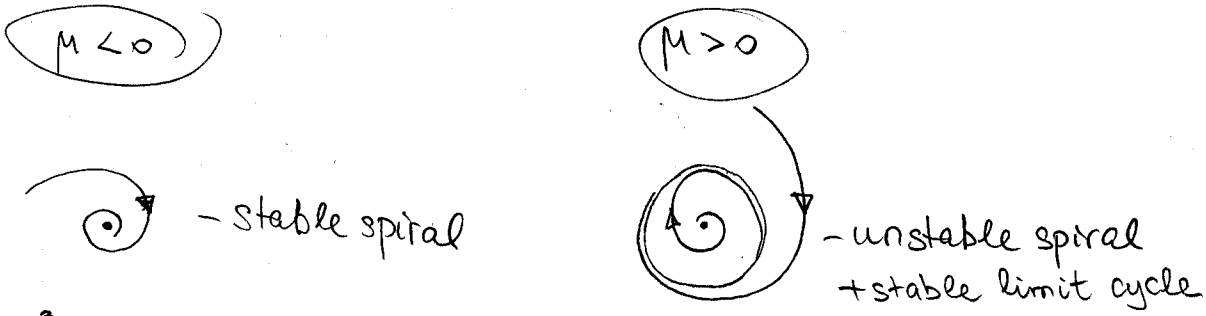


Both real



Complex conjugate

Supercritical Hopf:



- stable spiral

- unstable spiral + stable limit cycle

Example:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$

← 3 parameters:

μ – bifurcation (stability)

ω – frequency

b – amplitude/freq. dep.

↑

1-d pitchfork + rotation:

$\mu > 0$: stable f.p. $r = \pm\sqrt{\mu}$ → stable limit cycle, $r = \sqrt{\mu}$

$0 < \theta < 2\pi$

Eigenvalues:

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow \quad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta =$$

$$= (\mu - (x^2+y^2))x - (\omega + b(x^2+y^2))y =$$

$$= \mu x - \omega y + O(r^3)$$

Similarly $\dot{y} = \omega x + \mu y + O(r^3)$

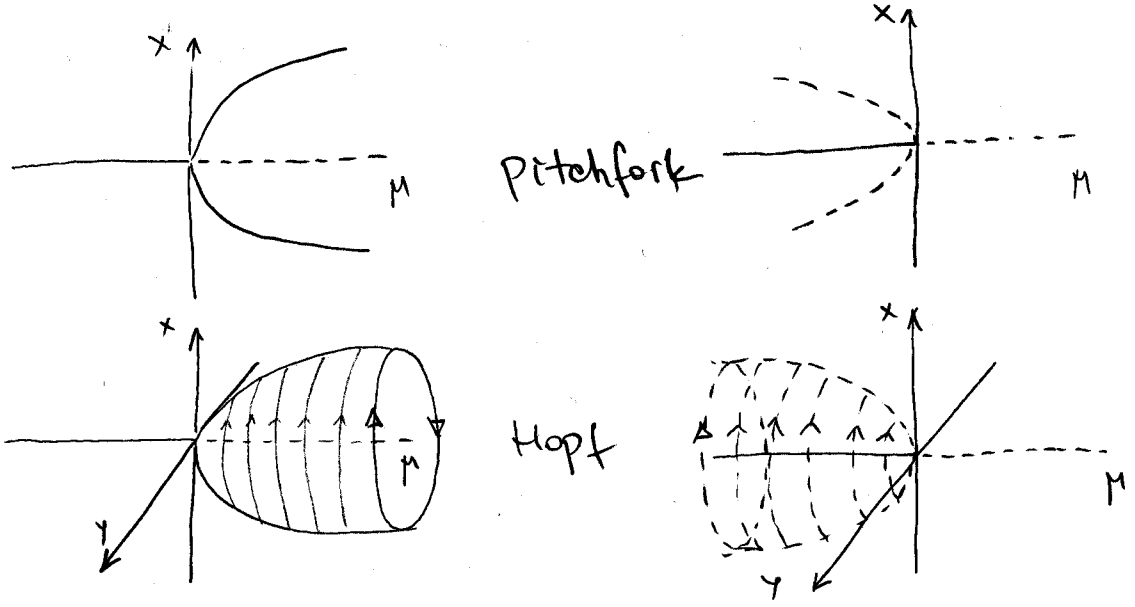
Jacobian: $A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \Rightarrow \quad \lambda = \mu \pm i\omega \rightarrow$ (un) stable spiral

Generically: near the bifurcation

a) Amplitude: $r \sim \sqrt{\mu - \mu_c}$

b) Frequency: $\omega = \mathrm{Im}\,\lambda + O(\mu - \mu_c)$

Subcritical Hopf



supercritical

subcritical

Can apply Poincare-Bendixson thm if fixed point is unstable:

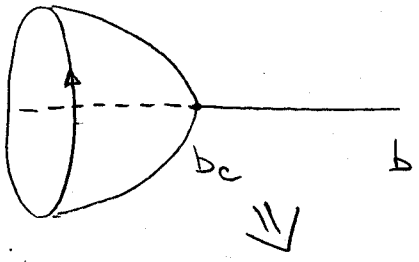
$$x^* = a/5 \quad , \quad y^* = 1+(x^*)^2 = 1+(a/5)^2$$

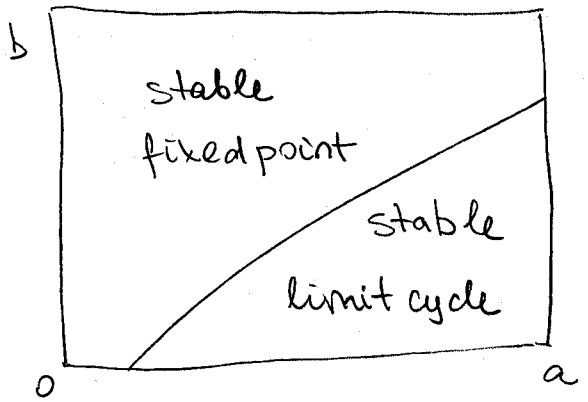
Jacobian:

$$\frac{1}{1+(x^*)^2}\begin{pmatrix} 3(x^*)^2-5 & -4x^* \\ 2b(x^*)^2 & -bx^* \end{pmatrix}$$

$$\Delta = \frac{5bx^*}{1+(x^*)^2} > 0 \Rightarrow \text{always a node}$$

$$\tau = \frac{3(x^*)^2-5-bx^*}{1+(x^*)^2} > 0 \Rightarrow \text{unstable} \quad \left(b < b_c = \frac{3}{5}a - \frac{25}{a}\right)$$





Supercritical Hopf!

Approximate period of oscillation:

For $b = b_c$, $\lambda = \pm i\sqrt{\Delta} = \pm i\left(\frac{15a^2-625}{a^2+25}\right)^{1/2}$

$$T = \frac{2\pi}{\mathrm{Im}\,\lambda} = 2\pi\left(\frac{15a^2-625}{a^2+25}\right)^{-1/2} + O(|b-b_c|)$$

$T \to \infty$, $a \to \sqrt{\frac{125}{3}}^{+}$

$T \to \frac{2\pi}{\sqrt{15}}$, $a \to \infty$