Global Bifurcations of Cycles

So far considered:

fixed points → fixed points

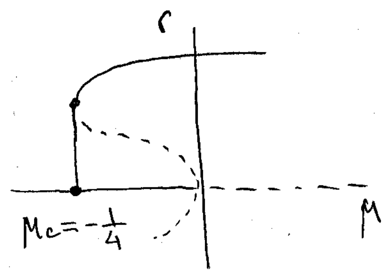
fixed points → limit cycles

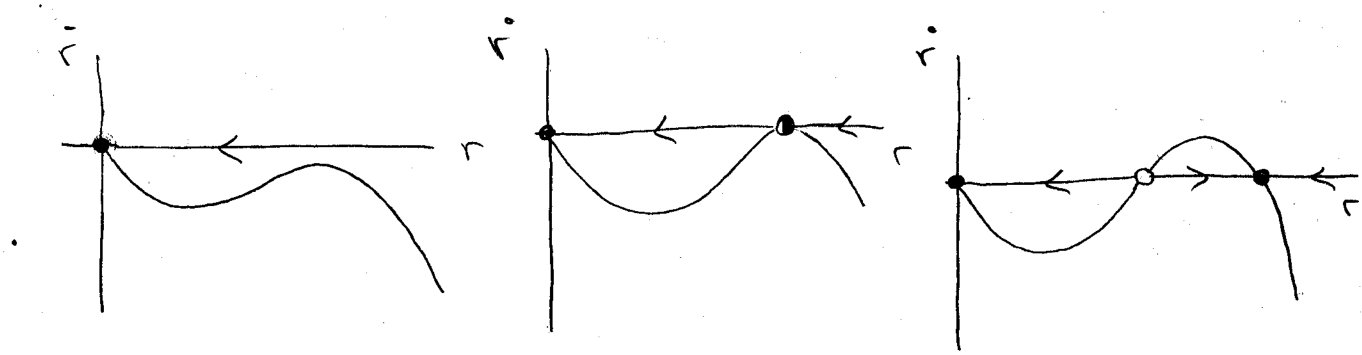
Now: what *else* can happen with limit cycles?

Limit cycles of *finite amplitude* can be created or destroyed.

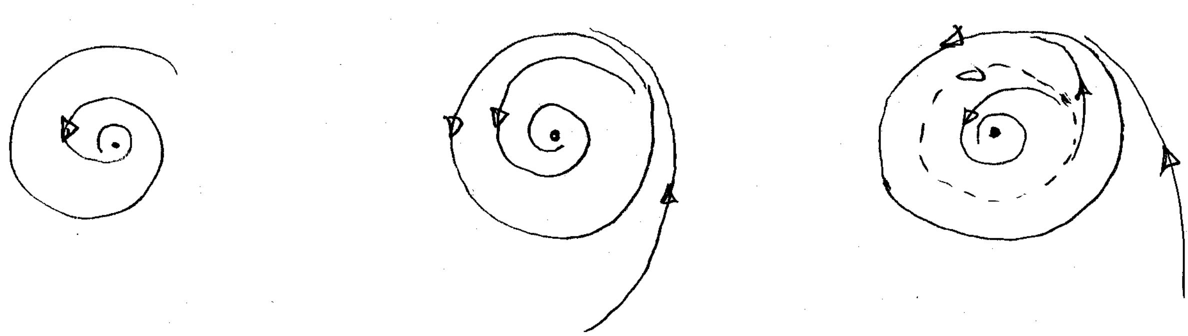
Saddle-Node Bifurcation of Cycles

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$





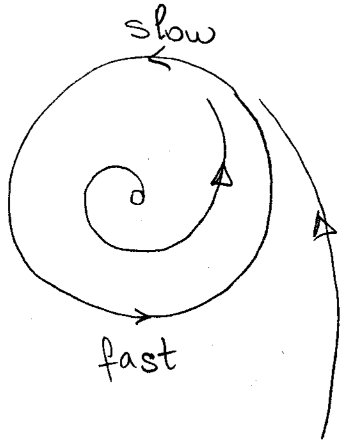
$\mu < \mu_c$ $\qquad$ $\mu = \mu_c$ $\qquad$ $0 > \mu > \mu_c$

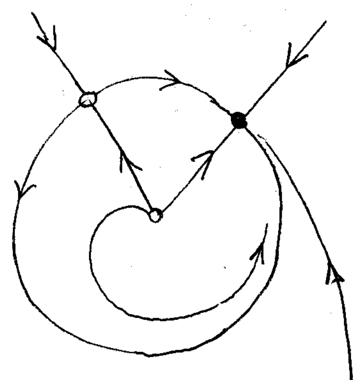


At the bifurcation:
$$\begin{cases} \text{Period: } O(1) \\ \text{Amplitude: } O(1) \end{cases}$$

Infinite Period Bifurcation

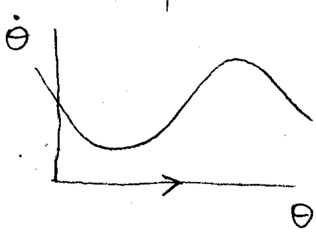
$$\begin{cases} \dot{r} = r(1-r^2) & \leftarrow \text{stable f.p. at } r=1 \\ \dot{\theta} = \mu - \sin\theta & \leftarrow \text{saddle-node bifuration at } \mu = \pm 1 \end{cases}$$

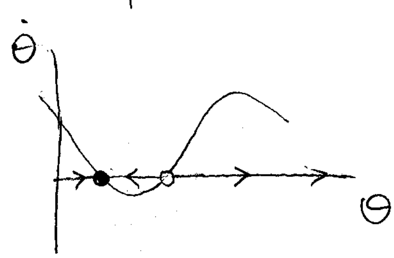




$\mu > 1$

$0 < \mu < 1$





Period: $T \sim |\mu - \mu_c|^{-1/2}$ (bottleneck)

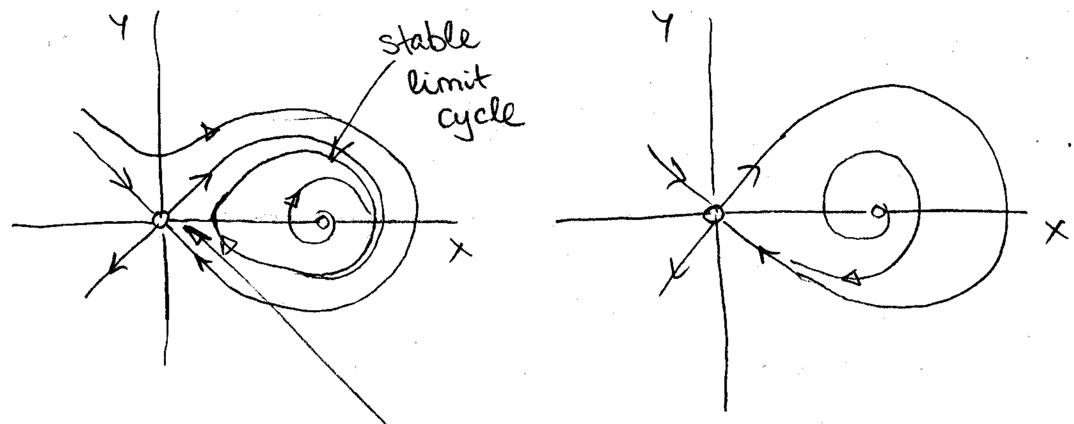
· Amplitude: $O(1)$

Homoclinic Bifurcation (Saddle-Loop)

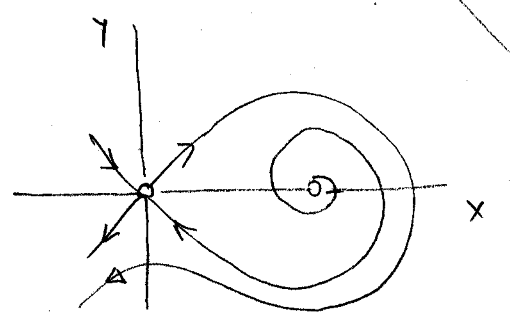
$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu x + x - x^2 + xy \end{cases}$$

Fixed points: $(0,0)$ - saddle

$(\mu+1, 0)$ - unstable spiral



$\mu < \mu_c$ $\mu = \mu_c$



$\mu > \mu_c$

Slow evolution near the saddle: time to get through the bottleneck: $t \sim -\ln|\mu-\mu_c|$

Amplitude : $r \sim O(1)$

Period : $T \sim -\ln|\mu-\mu_c|$

<u>Example:</u> $\ddot{x} + \varepsilon\dot{x}(x^2-1) + x = 0$ (Van der Pol)

At $\varepsilon=0$, $\ddot{x}+x=0 \Rightarrow \lambda_{1,2} = \pm i \Rightarrow$ Period $O(1)$

<u>Remember:</u> there is a <u>unique</u>, stable limit cycle with $r = 2 + O(\varepsilon)$ (courtesy of Lienard Theorem) $\Rightarrow$ Amplitude $O(1)$

$\Rightarrow$ Saddle-Node? No, there is no <u>unstable</u> limit cycle!

This is <u>Degenerate Hopf:</u> equations become conservative at the bifurcation point $\varepsilon=0$!

<u>Rescale:</u> $u = \sqrt{\varepsilon}x$

$\Rightarrow \ddot{u} + u + u^2\dot{u} - \varepsilon\dot{u} = 0 \Rightarrow u(t,\varepsilon) = 2\sqrt{\varepsilon}\cos t \Leftarrow$ Supercritical Hopf.

Scaling:

Important: can deduce the type of bifurcation without knowing dynamical equations.

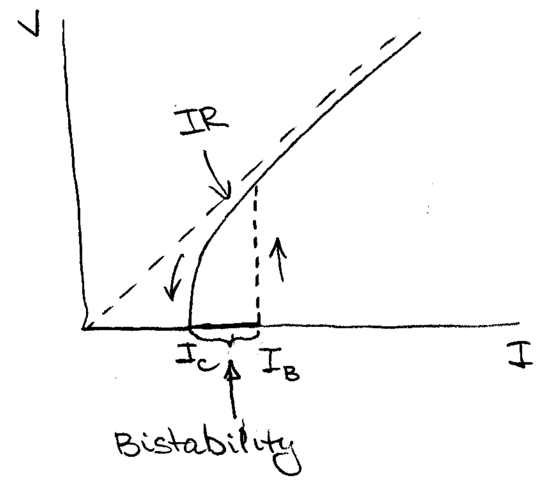
Useful for model validation!

Hysteresis in the Driven Pendulum / Josephson Junction

$$\frac{\hbar C}{2e}\ddot{\varphi} + \frac{\hbar}{2eR}\dot{\varphi} + I_c \sin\varphi = I_B$$

Voltage: $V = \frac{\hbar}{2e}\dot{\varphi}$

Current: $I = I_c \sin\varphi$



How does the bistability occur?

Nondimensionalize eq:

$$\tau = \left(\frac{2eI_c}{\hbar C}\right)^{1/2} t\,, \quad I = I_B/I_c\,, \quad \alpha = \left(\frac{\hbar}{2eI_cR^2C}\right)^{1/2}$$

$$\Rightarrow \varphi'' + \alpha\varphi' + \sin\varphi = I \quad \Rightarrow \begin{cases} \varphi' = y \\ y' = I - \sin\varphi - \alpha y \end{cases}$$

Fixed Points: $y^* = 0$, $\sin\varphi^* = I$

$\Rightarrow$ 2 f.p. for $I < 1$, (Saddle + Node)

1 f.p. for $I = 1$ (Saddle-Node bifurcation point)

0 f.p. for $I > 1$

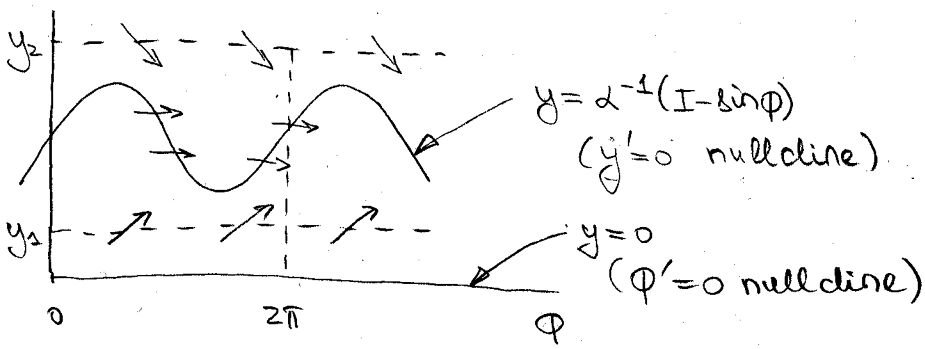
Jacobian: $A = \begin{pmatrix} 0 & 1 \\ -\cos\varphi^* & -\alpha \end{pmatrix}$

$$\begin{cases} \tau = -\alpha < 0 \\ \Delta = \cos\varphi^* = \pm\sqrt{1-I^2} \end{cases} \Rightarrow \begin{array}{l} \Delta = +\sqrt{1-\Delta^2} > 0 \Rightarrow \underline{\text{sink}} \\ \Delta = -\sqrt{1-\Delta^2} < 0 \Rightarrow \underline{\text{saddle}} \end{array}$$

The sink is a <u>node</u>, when $\tau^2 - 4\Delta = \alpha^2 - 4\sqrt{1-I^2} > 0$

otherwise it is a <u>spiral</u>

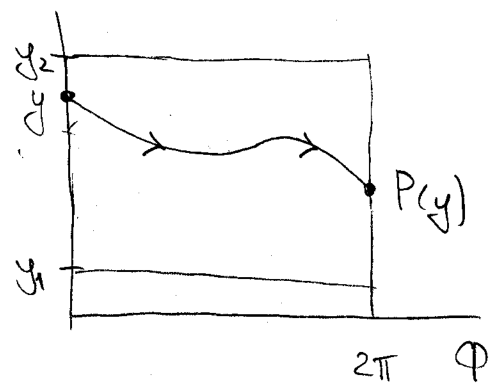
What happens for $I > 1$?

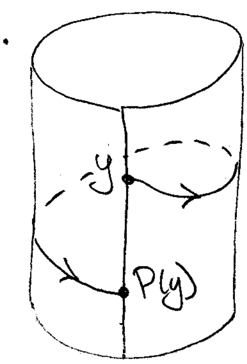


$\varphi' = y > 0$ - everywhere

Poincare section:

$$P(y) = y(2\pi)\big|_{y(0)=y}$$





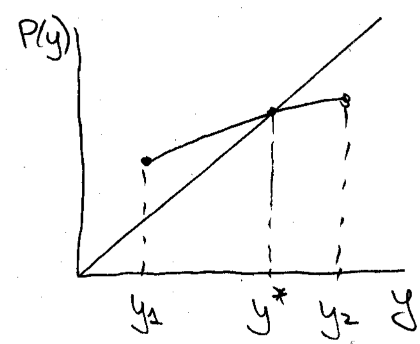
$P(y)$ - first return map to $\varphi (\mathrm{mod}\, 2\pi) = 0$.

Observe:

a) $P(y_1) > y_1$, $P(y_2) < y_2$

b) $P(y)$ - continuous

c) $P(y)$ - monotonic

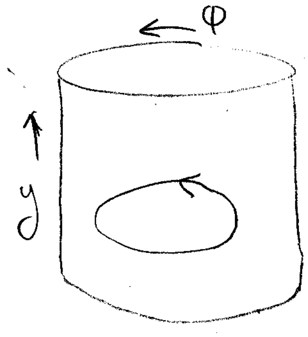


$\Rightarrow \exists y^* : P(y^*) = y^*$

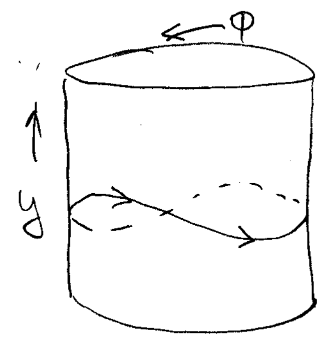
Fixed Point of the map ⇒ Periodic orbit

isolated fixed point → limit cycle

<u>Uniqueness</u>

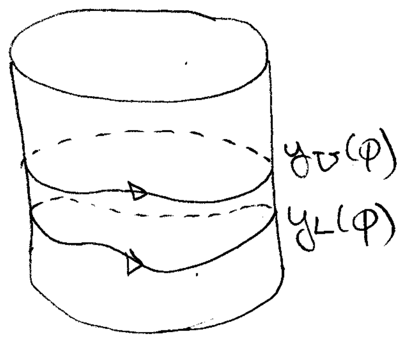


libration ($I<1$)



rotation ($I>1$)

Suppose there are two limit cycles:



can't cross ⇒ $y_U(\varphi) > y_L(\varphi)$

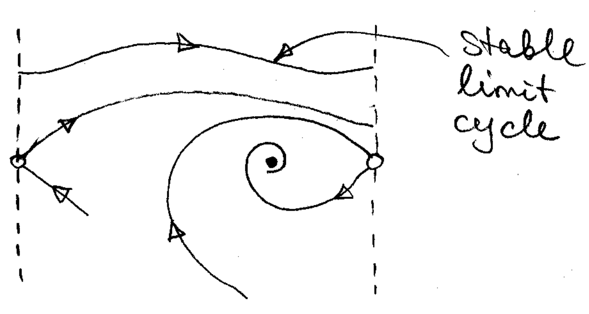
<u>Energy</u>: $E = \frac{1}{2}y^2 - \cos\varphi = \frac{1}{2}\varphi'^2 - \cos\varphi$

$$0 = \Delta E = \int_0^{2\pi} \frac{dE}{d\varphi}\, d\varphi = \int_0^{2\pi} \left(y\frac{dy}{d\varphi} + \sin\varphi \right) d\varphi =$$

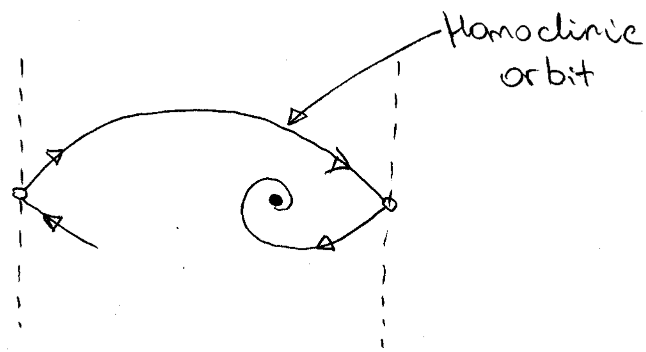
$$= \int_0^{2\pi} \left(\frac{dy/d\tau}{d\varphi/d\tau} \cdot y + \sin\varphi \right) d\varphi = \int_0^{2\pi} \left(I - \alpha y - \cancel{\sin\varphi} + \cancel{\sin\varphi} \right) d\varphi$$

$$\Rightarrow \int_0^{2\pi} y\, d\varphi = \frac{2\pi I}{\alpha} \quad : \quad \text{But } \int_0^{2\pi} y_U\, d\varphi > \int_0^{2\pi} y_L\, d\varphi \rightarrow \text{contradiction!}$$

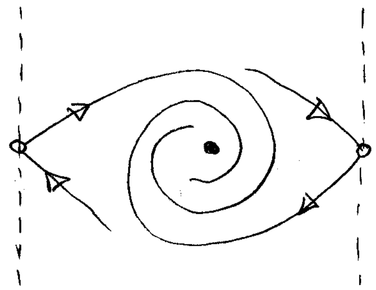
Follow the pendulum as it goes over the top ($I \downarrow$):



$I_c < I < 1$



$I = I_c$



Homoclinic Bifurcation

⇒ (for small α)

$I < I_c$

(As torque is decreased it takes more and more time to get over the top, until the torque is too small to get past the saddle point.)