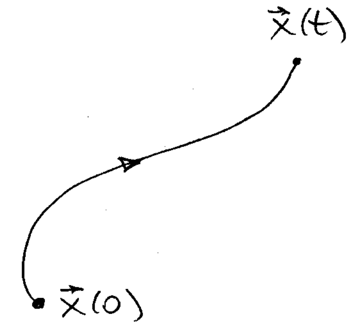
Continuous-Time Dynamical Systems

Abstract definitions: $\dot{\vec{x}} = \vec{f}(\vec{x}, t)$



1) $\vec{x}$ – state of a system
2) $\vec{x} \in M$ – phase space
3) Vector field (flow) $\vec{f}$:

   $\vec{f}: M \to M$ – autonomous

   $\vec{f}: M \times T \to M$ – non-autonomous
4) (M, f) define a dynamical system
5) $\vec{x}(t)$ – trajectory of the dynamical system

Autonomous system:

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \quad \vec{x} = (x_1, \dots, x_n) \quad \text{– n-dimensional}$$

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n) \\ \cdots \\ \dot{x}_n = f_n(x_1, \dots, x_n) \end{cases} \quad \text{– system of } n \text{ equations}$$

Non-autonomous system:

- Solution depends on n initial values $x_1(0), \dots, x_n(0)$ plus time t $\Rightarrow$ $n+1$-dimensional system:

$$\dot{\vec{x}} = \vec{f}(\vec{x}, t) \qquad t \equiv x_{n+1}$$

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n, x_{n+1}) \\ \cdots \\ \dot{x}_n = f_n(x_1, \dots, x_n, x_{n+1}) \\ \dot{x}_{n+1} = 1 \end{cases} \quad \text{– system of } n+1 \text{ equations}$$

Flows on the Line

Simplest case: $n=1$

Terminology: first order or one-dimensional system

Think Geometrically:

$\dot{x} = \sin x \rightarrow$ velocity $-1 < \dot{x} < 1$, dependent on x

Rare case – can solve analytically:

$$dt = \frac{dx}{\sin x} = \csc x \, dx$$

Integrate: $t = \int \csc x \, dx = -\ln|\csc x + \cot x| + C$

Initial condition: $x(0) = x_0 \Rightarrow t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$

Difficult to interpret! Try to answer these questions:

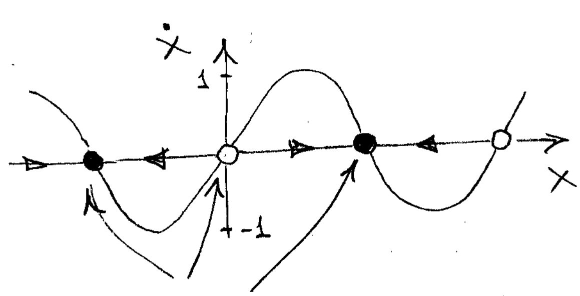
1. Given $x_0 = \frac{\pi}{4}$, what is $x(t)$ as $t \rightarrow \infty$?
2. Depending on x_0, what kind of solutions are there qualitatively?

Geometrically:

$\dot{x} = f(x)$ — vector field or flow

○ unstable, repelling, source

● stable, attracting, sink

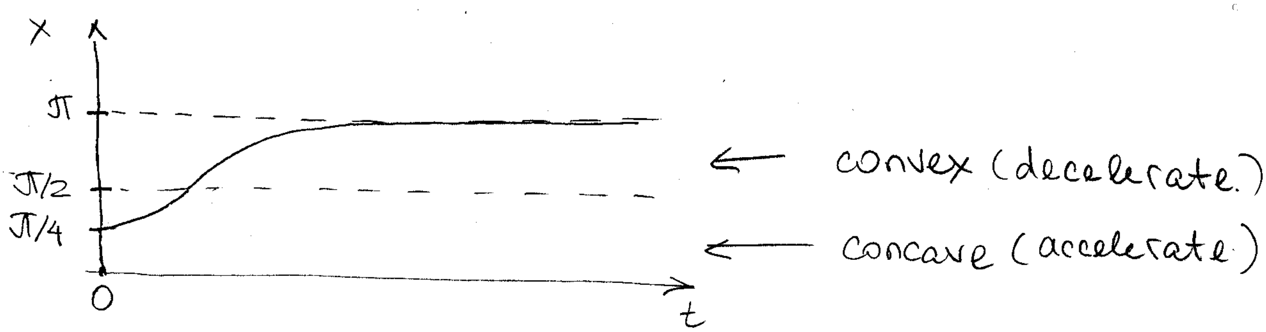


Equilibrium (stagnation, fixed) points: $\dot{x} = f(x) = 0$

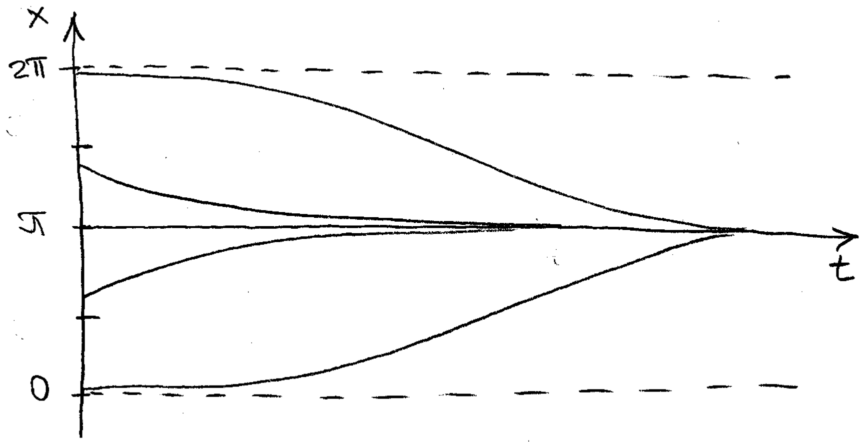
↑ asymptotic ($t \rightarrow \infty$) state of the system

Answers:

1.



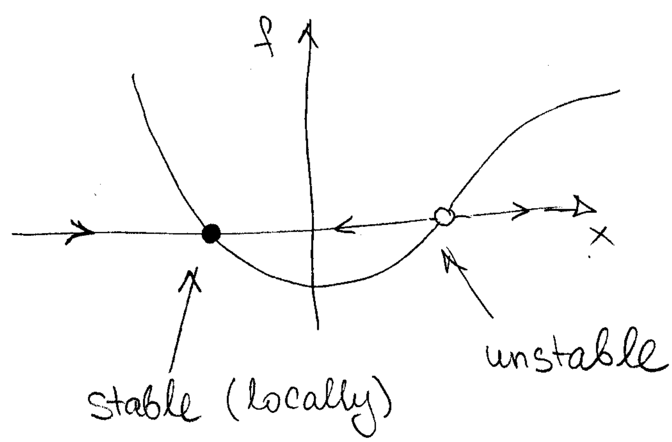
2.

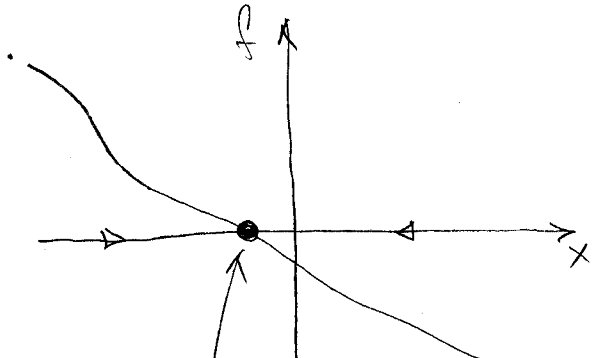


periodic in x with period 2π

Fixed Points and stability

Arbitrary f:

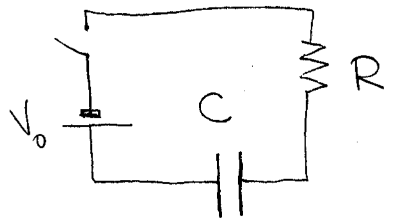




Mechanical analogy:

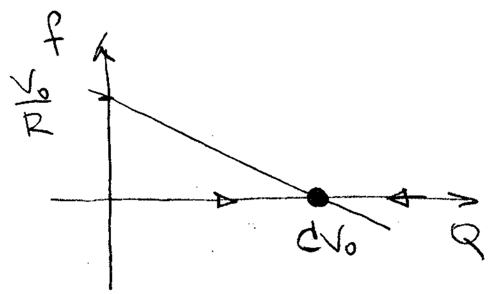
overdamped motion in a potential field

RC circuit

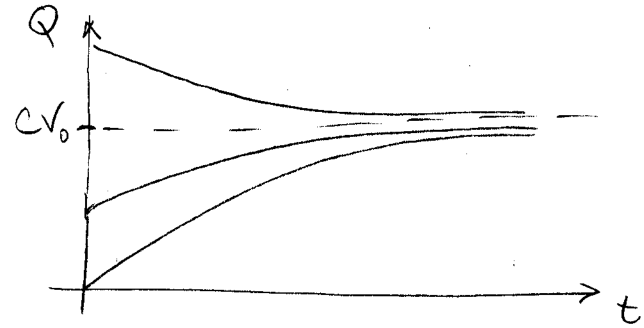


$$-V_0 + RI + \frac{Q}{C} = 0$$

$$I = \dot{Q} \quad \Rightarrow \quad \dot{Q} = f(Q) = \frac{V_0}{R} - \frac{1}{RC}Q$$

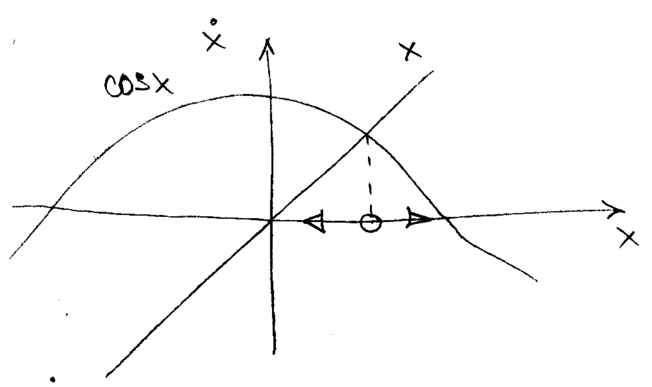


$Q^* = CV_0$ - globally stable fixed point



Example:

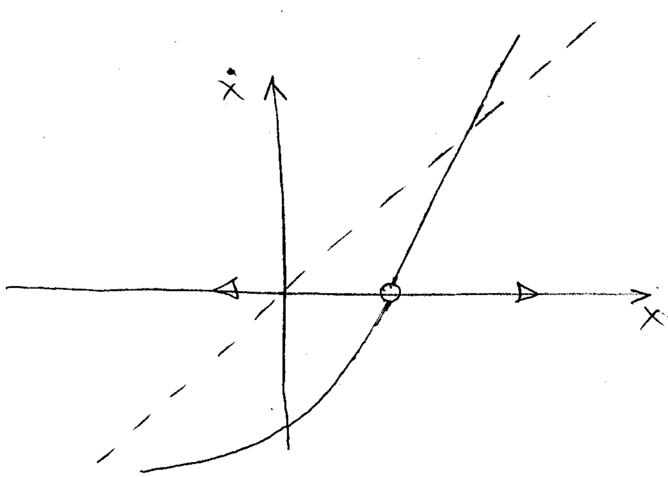
$\dot{x} = x - \cos x \qquad f(x) = x - \cos x$



$$\dot{x} = x^* - \cos x^* = 0$$

$$\Rightarrow \underline{x^* = \cos x^*}$$

single fixed point,
<u>unstable</u>.



Population Dynamics

Individual: probability (in a unit of time) to

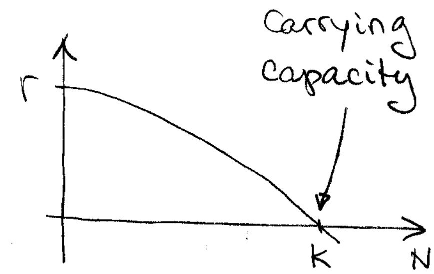
produce one offspring: p_b

die: p_d

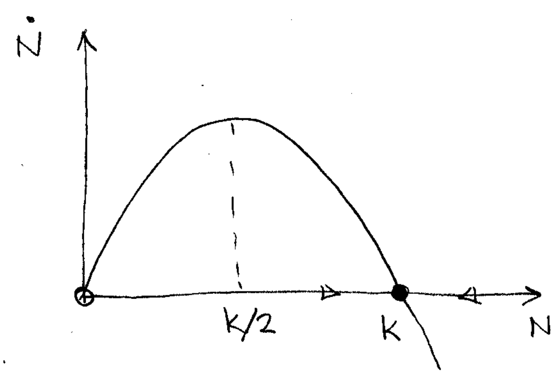
$$\Rightarrow \quad \dot{N} = \underset{\text{birthrate}}{bN} - \underset{\text{deathrate}}{dN} = \underbrace{(b-d)}_{r} N = \underset{\text{growth rate}}{r} N \leftarrow \text{number of individuals}$$

Interaction: (overcrowding, limited resources)

$$\frac{\dot{N}}{N} = r - \frac{r}{K} N \quad \text{(Verhulst, May)}$$

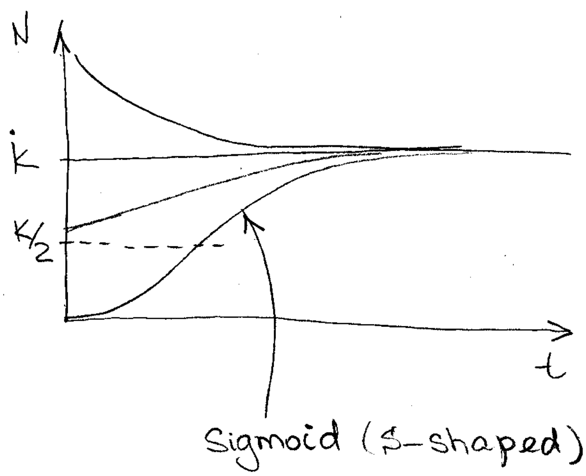


Logistic equation: $\dot{N} = rN\left(1 - \frac{N}{K}\right)$



$N^* = K$ - stable point

$N = 0$ - unstable point



Agreement with experiment:

- yeast, bacteria – <u>excellent</u> (simple life cycle)
- fruit flies, flour beetles – <u>bad</u> (complex life cycle: eggs, larvae $\Rightarrow$ time delays)

Linear Stability Analysis

Quantitative information about the rate of growth / decay.

Let x^* – fixed point

$\eta(t) = x(t) - x^*$

$$\Rightarrow \dot{\eta} = \dot{x} = f(x^* + \eta) = \overset{0}{\overset{\shortparallel}{f(x^*)}} + \eta f'(x^*) + O(\eta^2)$$

$$\boxed{\dot{\eta} = f'(x^*)\,\eta + O(\eta^2)}$$

1) $f'(x^*) \neq 0$: the sign and magnitude of $f'(x^*)$ determine the stability and growth / decay rate.

$f'(x^*) > 0$: unstable – perturbation grows exponentially,

$f'(x^*) < 0$: stable – perturbation decays exponentially.

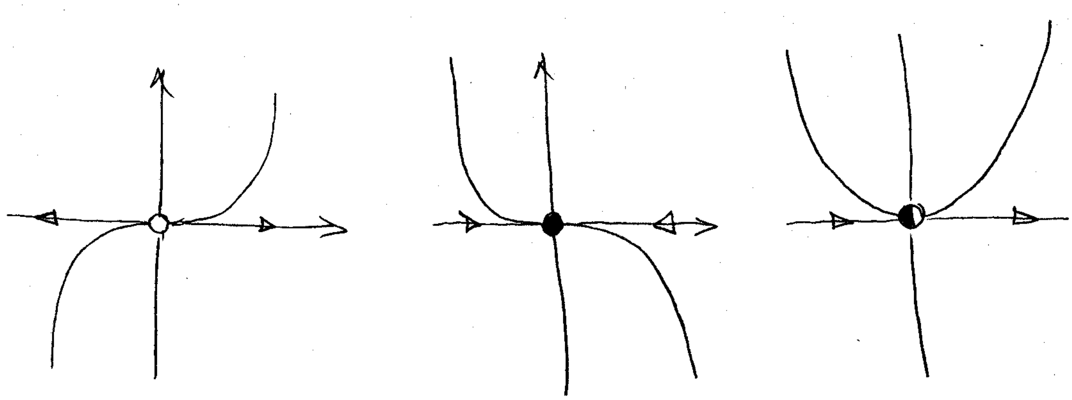
Characteristic time scale for evolution near x^*: $[f'(x^*)]^{-1}$.

Example: $\dot{x} = \sin x \Rightarrow f'(x^*) = \cos n\pi = \begin{cases} 1, & n\text{ – even (unstable)} \\ -1, & n\text{ – odd (stable)} \end{cases}$

2) $f'(x^*) = 0$: Nothing can be said in general!

Examples:

$\dot{x} = x^3$ $\quad\quad$ $\dot{x} = -x^3$ $\quad\quad$ $\dot{x} = x^2$



More Weird Cases: Existence & Uniqueness

Example: $\dot{x} = x^{1/3}$, $x_0 = 0$

$\boxed{x=0}$ – solution, but:

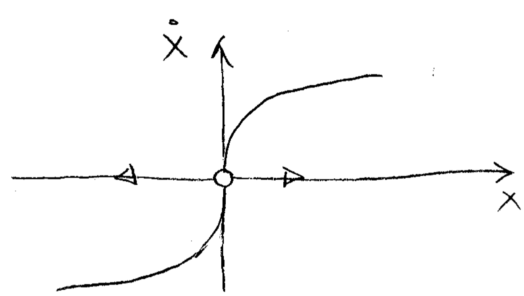
$$\frac{dx}{x^{1/3}} = dt \Rightarrow \int x^{-1/3}dx = \frac{3}{2}x^{2/3} = \int dt = t + C$$

$$x(0) = x_0 = 0 \Rightarrow C = 0 \Rightarrow \boxed{x = \left(\frac{2}{3}t\right)^{3/2}}$$ – also solution!

In fact, there are infinitely many solutions!

$x^* = 0$ – very unstable:

$f'(0) = +\infty$



Uniqueness & Existence theorem:

Given initial value problem $\dot{x} = f(x)$, $x(0) = x_0$, such that $f(x), f'(x) \in C^0$ on an open interval R: $x_0 \in R$ there exist a unique solution $x(t)$ for $t \in (-\tau, \tau)$.

Finite-time singularity

$\dot{x} = 1 + x^2$, $x(0) = 0$

$$\int \frac{dx}{1+x^2} = \int dt \Rightarrow \arctan x = t + \overset{=0}{C} \Rightarrow x(t) = \tan t.$$

Solution exists for $-\frac{\pi}{2} < t < \frac{\pi}{2}$

At $t = \frac{\pi}{2}$ finite time singularity occurs:

Solution blows up due to extremely fast growth