Impossibility of Oscillations

Types of behavior in first order systems:

- Trajectory approaches a fixed point
- Trajectory diverges to $\pm\infty$

$\Rightarrow$ Can increase / decrease monotonically or remain constant

(The direction of the flow never reverses: $f(x)$)

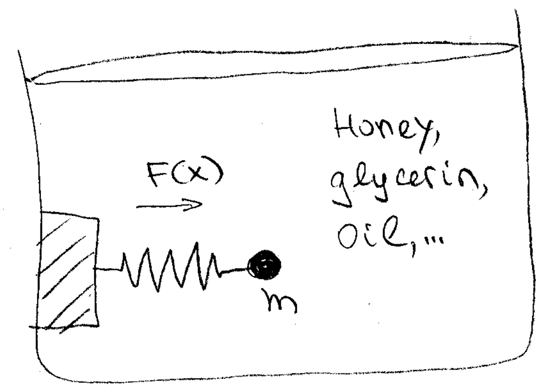
$\Rightarrow$ No periodic solutions to $\dot{x} = f(x)$.

Exception: flow on a circle

Mechanical Analog

Overdamped pendulum:

$$m\ddot{x} - \gamma\dot{x} + F(x) = 0$$



For $\gamma\dot{x} \gg m\ddot{x}$, $\dot{x} \approx \frac{1}{\gamma} F(x)$

$\Rightarrow$ Exponential decay to equilibrium point.

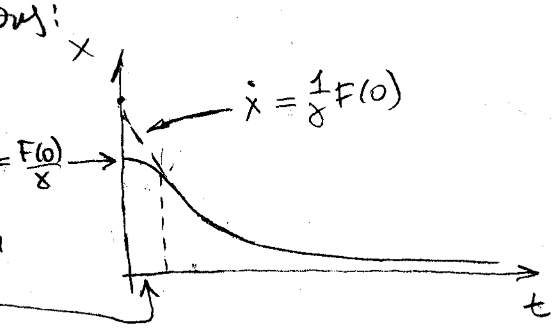
Fine point: Small coefficient at highest derivative

$\dot{x} = \frac{1}{\gamma} F(x)$ – one initial condition:

$$x(0) = x_0$$

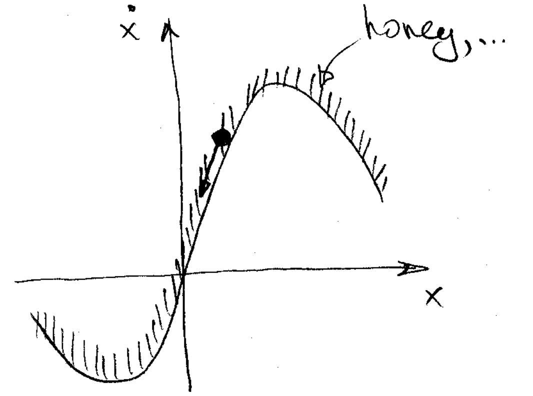
$m\ddot{x} - \gamma\dot{x} + F(x) = 0$ – two initial conditions:

$$x(0) = x_0, \quad \dot{x}(0) = V_0$$



"Adjust" to second i.c. in a "boundary layer"

Potential



$\dot{x} = f(x)$, $f(x) = -\dfrac{dV}{dx}$

$V(x)$ – potential

$V = V(x(t))$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt} = -\frac{dV}{dx}\cdot\frac{dV}{dx} = -\left(\frac{dV}{dx}\right)^2 \le 0$$

Lyapunov function:

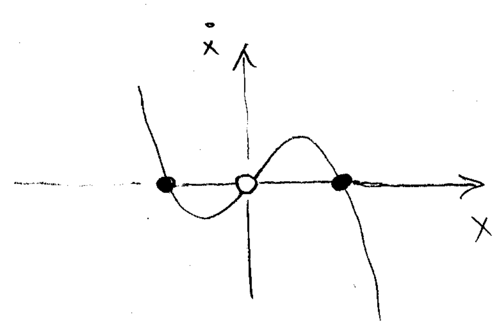
- $\dfrac{dV}{dt} \le 0 \Rightarrow V$ – decreases monotonically with time
- $\left.\dfrac{dV}{dx}\right|_{x=x^*} = 0 \Rightarrow x^*$ – equilibrium point, $f(x^*) = 0$.

Lyapunov theorem:

If there exists $V(x) \in C^1(x^*-\delta, x^*+\delta)$ such that

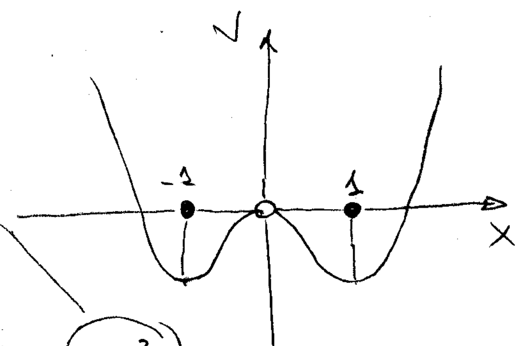
1) $V(x) > V(x^*)$ for $\forall x \in (x^*-\delta, x^*) \cup (x^*, x^*+\delta)$
2) $\dfrac{dV}{dt} \le 0$, $t \ge t_0$

then x^* is stable



$$\left[\begin{array}{l} \delta < \infty \ \text{– local stability} \\ \delta = \infty \ \text{– global stability} \end{array}\right.$$

Example:



$\dot{x} = x - x^3$

$\Rightarrow V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C \quad (C=0)$

Spontaneous symmetry breaking due to $-x^3$

Higher order systems:

Example: Second derivative

$$\ddot{x} + \gamma \dot{x} + \frac{g}{L}\sin x = 0 \quad \leftarrow \text{Second order ODE}$$

Change of variables: $\dot{x} = v$

$$\begin{cases} \dot{v} = -\gamma v - \frac{g}{L}\sin x \\ \dot{x} = v \end{cases} \quad \leftarrow \text{system of } \underline{2} \text{ first order ODEs}$$

More generally:

$$x^{(N)} + F(x^{(N-1)}, x^{(N-2)}, \dots, \ddot{x}, \dot{x}, x) = 0 \leftarrow N^{th} \text{ order ODE}$$

Change of variables: $x^{(N-1)} = v_N, \dots, \ddot{x} = v_3, \dot{x} = v_2, x = v_1$

$$x^{(N)} = \frac{d}{dt}x^{(N-1)} = \dot{v}_N = -F(x^{(N-1)}, \dots, x) = -F(v_N, \dots, v_1)$$

$$\Rightarrow \begin{cases} \dot{v}_N = -F(v_N, \dots, v_1) \\ \dot{v}_{N-1} = v_N \\ \dots \\ \dot{v}_1 = v_2 \end{cases} \quad \leftarrow \text{system of } \underline{N} \text{ first order ODEs}$$

Numerical integration:

A dynamical system is generically described by

1) a system of $\underline{N}$ (nonautonomous) ODEs:

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n, t), \quad i = 1, \dots, N$$

2) a set of $\underline{N}$ boundary (initial) conditions

$$x_i(t_i) = x_i^0, \quad i = 1, \dots, N$$

Forward (Explicit) Euler method

$$\dot{x} = f(x,t)$$

"Undo" the derivative:

$$\dot{x} = \frac{dx}{dt} = \lim_{dt\to 0} \frac{x(t+dt)-x(t)}{dt} \approx \frac{x(t+dt)-x(t)}{dt}, \quad dt\text{-small}$$

$$\frac{x(t+dt)-x(t)}{dt} = \frac{\left(x(t)+\dot{x}(t)dt+\frac{1}{2}\ddot{x}(t)dt^2+\frac{1}{3!}\dddot{x}(t)+\dots\right)-x(t)}{dt} =$$

$$= \dot{x}(t)+\frac{1}{2}\ddot{x}(t)dt+\frac{1}{3!}\dddot{x}(t)dt^2+\dots = \dot{x}(t)+O(dt^2)$$

$\Rightarrow$ first order correct (in dt)

Integration rule:

$$\frac{x(t+dt)-x(t)}{dt} = \dot{x}(t)+O(dt^2) \approx f(x(t),t)$$

$$\Rightarrow x(t+dt) = x(t)+dt\,f(x(t),t)$$

Problems:
- Only first order accurate (requires very small steps to achive good accuracy)
- unstable for $dt > dt^*$

Let $x(t+n\cdot dt) \equiv x^n$:

$$x^n = \bar{x}^n + \eta^n$$ exact solution

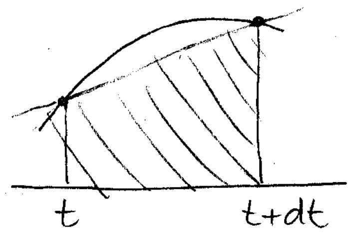
$$\bar{x}^{n+1}+\eta^{n+1} = x^{n+1} = x^n + dt\,f(x^n) = \bar{x}^n+\eta^n+dt\,f(\bar{x}^n+\eta^n) =$$
$$= \bar{x}^n+\eta^n+dt\,f(\bar{x}^n)+\eta^n dt\,f'(\bar{x}^n)$$

$$\bar{x}^{n+1} = \bar{x}^n + dt\,f(\bar{x}^n) + O(dt^2)$$

$$\Rightarrow \eta^{n+1} = (1+dt\,f'(\bar{x}^n))\,\eta^n$$

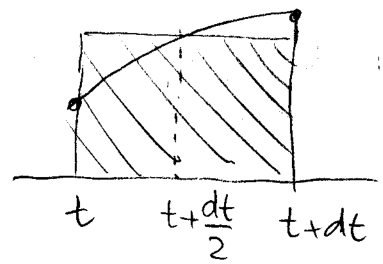
Second order methods:

Improved Euler method:



$$\begin{cases} \tilde{x}(t+dt) = x(t) + f(x(t),t)\,dt & \text{– trial step} \\ x(t+dt) = x(t) + \frac{1}{2}\left[f(x(t),t) + f(\tilde{x}(t+dt), t+dt)\right]dt & \text{– real step} \end{cases}$$

Second order Runge-Kutta:



$$\begin{cases} \tilde{x}(t+\frac{1}{2}dt) = x(t) + \frac{1}{2} f(x(t),t)\,dt \\ x(t+dt) = x(t) + f(\tilde{x}(t+\frac{1}{2}dt), t+\frac{1}{2}dt)) \end{cases}$$

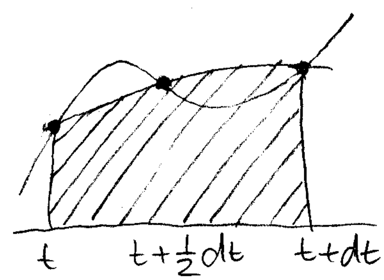
$$x(t+dt) - x(t) = dt\,\dot{x}(t) + \frac{1}{2}dt^2\ddot{x}(t) + o(dt^3)$$

$$\dot{x} = f(x,t) \Rightarrow \frac{d\dot{x}}{dt} = \ddot{x} = \left[\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}\right]_{x,t} \Rightarrow$$

$$x(t+dt) - x(t) = dt\,f + \frac{1}{2}dt^2\left[\frac{\partial f}{\partial x}f + \frac{\partial f}{\partial t}\right] + o(dt^3)$$

$$f(x(t) + \frac{1}{2}f(x(t),t)\,dt,\ t+\frac{1}{2}dt)\,dt$$
$$= \left[f + \frac{1}{2}dt\left[\frac{\partial f}{\partial x}\cdot f + \frac{\partial f}{\partial t}\right]_{x,t} + o(dt^2)\right]dt$$

Fourth order Runge-Kutta:



$$k_1 = f(x(t),t)\,dt$$
$$k_2 = f(x(t+\frac{1}{2}k_1), t+\frac{1}{2}dt)$$
$$k_3 = f(x(t+\frac{1}{2}k_2), t+\frac{1}{2}dt)$$
$$k_4 = f(x(t+k_3), t+dt)$$

$$x(t+dt) = x(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Final note: Beware of taking dt too small:

digital arithmetics $\Rightarrow$ roundoff error $\sim \begin{cases} 10^{-8} & \text{– single precision} \\ 10^{-16} & \text{– double precision} \end{cases}$