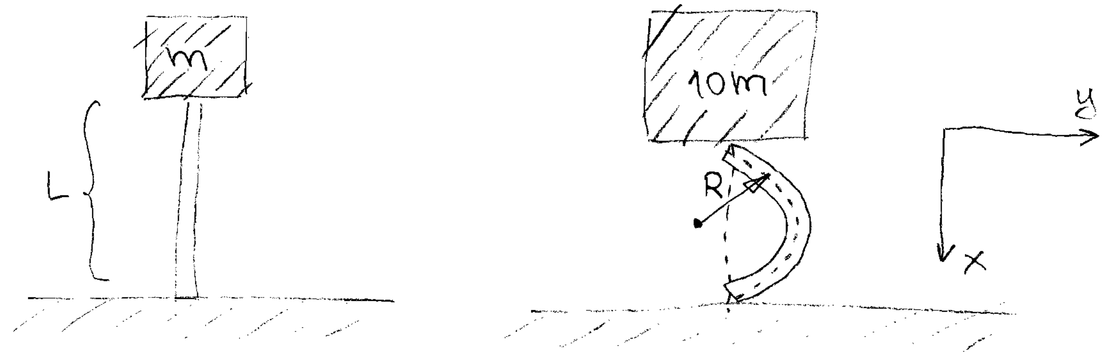
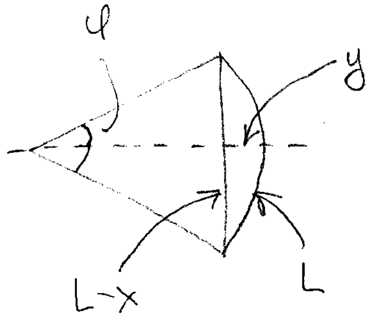
Spontaneous Symmetry Breaking - Buckling



Potential energy: $U = mg(L-x)$ - gravitational

$+ k_1 x^2$ - compression

$+ k_2 \left(\frac{1}{R}\right)^2$ - bending



Assume small deformations
$(x \ll L \Rightarrow \varphi \ll 1)$

$$\left.\begin{array}{l} L-x = 2R\sin\frac{\varphi}{2} \\ L = R\varphi \end{array}\right\} \Rightarrow L-x = 2R\sin\frac{L}{2R} \approx 2R\left(\frac{L}{2R} - \frac{1}{3!}\left(\frac{L}{2R}\right)^3\right) = L - \frac{L^3}{24R^2}$$

$$\Rightarrow x \approx \frac{L^3}{24R^2} \qquad \left(\frac{1}{R} \approx \left(\frac{24x}{L^3}\right)^{1/2}\right)$$

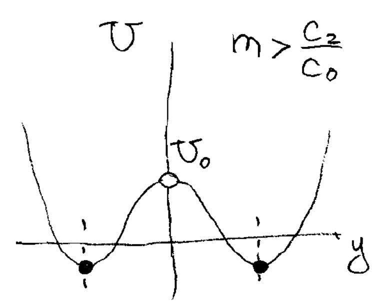
$$y = R(1-\cos\tfrac{\varphi}{2}) \approx R\left(\tfrac{1}{8}\varphi^2\right) = \tfrac{1}{2}R\left(\tfrac{L}{2R}\right)^2 = \tfrac{1}{8}\tfrac{L^2}{R} \approx \left(\tfrac{3}{8}Lx\right)^{1/2}$$

The stable equilibrium is defined by the minimum of potential energy:

$$U(y) = mgL - mg\,x(y) + k_1 x(y)^2 + k_2 \frac{1}{R(y)^2} =$$

$$= U_0 - m\cdot c_0 y^2 + c_1 y^4 + c_2\cdot y^2$$

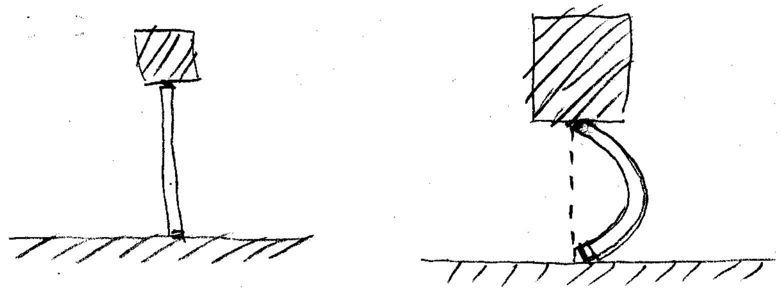
$$= U_0 + (c_2 - m c_0)y^2 + c_1 y^4$$



Bifurcations in 1D

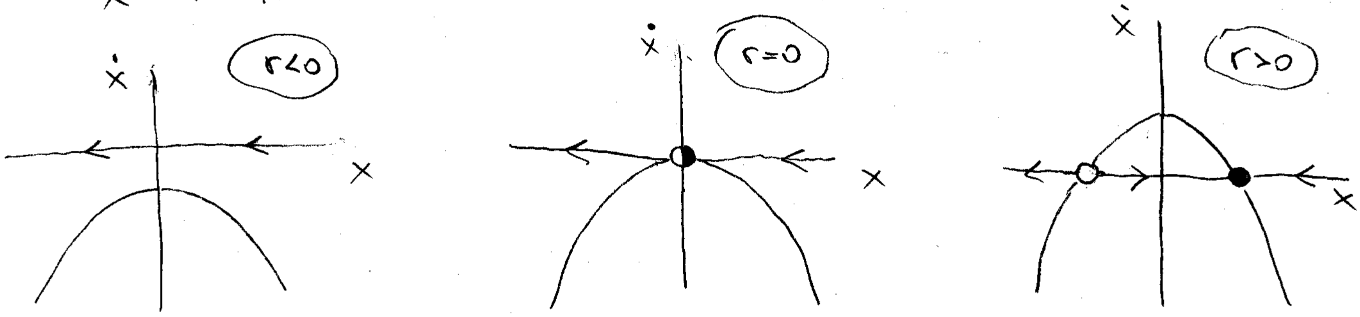
$\dot{x} = f(x, r)$ — r: parameter (load)

Buckling:



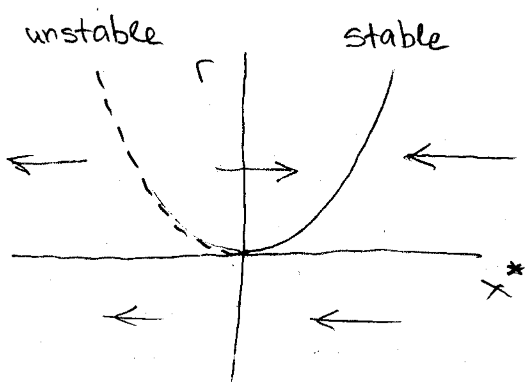
Saddle-Node bifurcation:

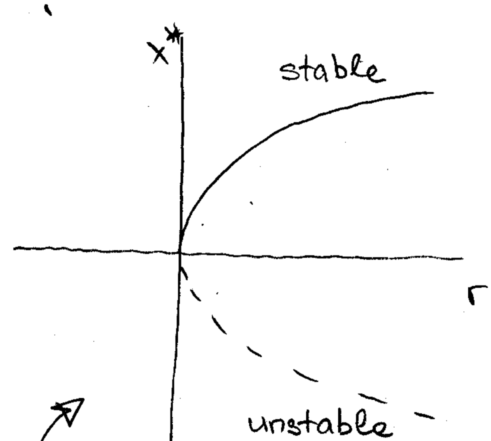
$\dot{x} = r - x^2$



- At $r=0$ a pair of fixed points ($x^* = \pm\sqrt{r}$) are created
- Bi-furcation = two branches of solution are created

Graphical representation:





1) "fold" bifurcation
2) "turning point"
3) "blue sky" bifurcation
4) "saddle-node" → will explain later.

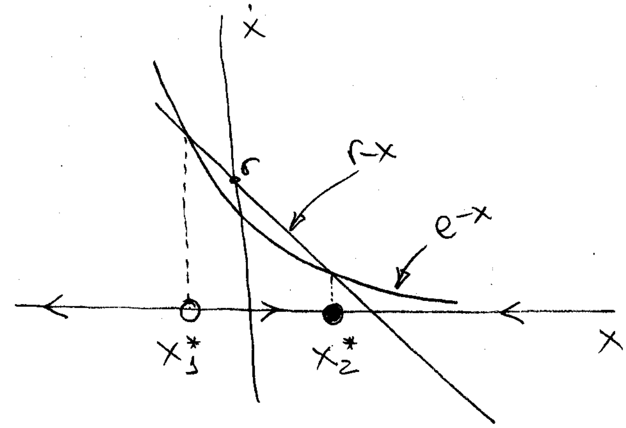
Bifurcation diagram

Example:

$$\dot{x} = r - x - e^{-x}$$

Fixed points: $r = x^* + e^{-x^*}$ – transcendental equation on $x^*(r)$

use geometrical approach:

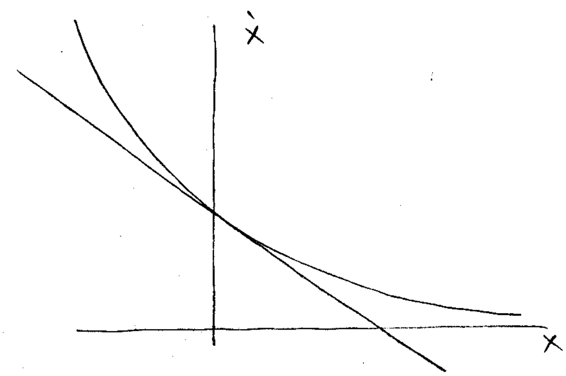


At the bifurcation point:

a) touch: $r - x = e^{-x}$

b) same slope: $-1 = -e^{-x}$

$\Rightarrow x^* = 0, \; r_c = 1$



$f(x,r) = r - x - e^{-x}$ – Taylor expand around (x^*, r_c):

$$\dot{x} = (r-1) + \cancel{1} - x - \cancel{1} + x - \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots = (r-1) - \frac{1}{2}x^2 + \dots$$

Rescale: $\begin{cases} x \to x' = \frac{1}{2}x & (x = 2x') \\ r \to r' = \frac{1}{2}(r-1) & (r = 2r'+1) \end{cases}$

$$2\dot{x}' = 2r' - 2x'^2 \Rightarrow \underline{\underline{\dot{x}' = r' - x'^2}}$$

generic (normal) form.

Linear stability analysis:

$$\dot{x} = f(x) = r - x^2, \quad x^* = \pm\sqrt{r}$$

$$f'(x) = -2x = \mp 2\sqrt{r} \Rightarrow \begin{cases} x^* = \sqrt{r},\ f'(x^*) < 0 - \text{stable} \\ x^* = -\sqrt{r},\ f'(x^*) > 0 - \text{unstable} \end{cases}$$

Normal Forms (Poincaré, turn of the last century)

$\dot{x} = r - x^2$ is a typical representative of all saddle-node bifurcations.

$\dot{x} = r + x^2$ is another typical representative obtained by $x \to -x$, $r \to -r$

More generally, Taylor expansion at the bifurcation point:

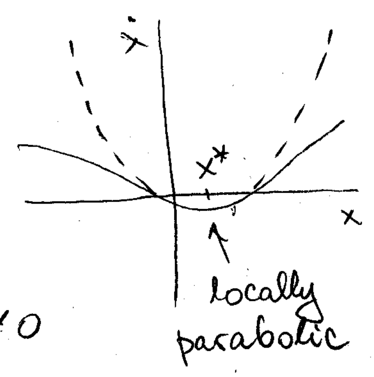
$$\dot{x} = f(x,r) = f(x^*, r_c) + (x - x^*)\frac{\partial f}{\partial x}\Big|_{x^*, r_c} + (r - r_c)\frac{\partial f}{\partial r}\Big|_{x^*, r_c}$$

$$+ \frac{1}{2}(x - x^*)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{x^*, r_c} + (x - x^*)(r - r_c)\frac{\partial^2 f}{\partial x \partial r}\Big|_{x^*, r_c} + \frac{1}{2}(r - r_c)^2 \frac{\partial^2 f}{\partial r^2}\Big|_{x^*, r_c} + \dots$$

For any Saddle-Node bifurcation:

1) $f(x^*, r_c) = 0$ (x^* is a fixed point)

2) $\frac{\partial f(x^*, r_c)}{\partial x} = 0$ ($f(x)$ is tangential to $x = 0$ at $x = x^*$ for saddle-node bifurcations)



In addition, generically $\frac{\partial f}{\partial r}\Big|_{x^*, r_c} \neq 0$, $\frac{\partial^2 f}{\partial x^2}\Big|_{x^*, r_c} \neq 0$

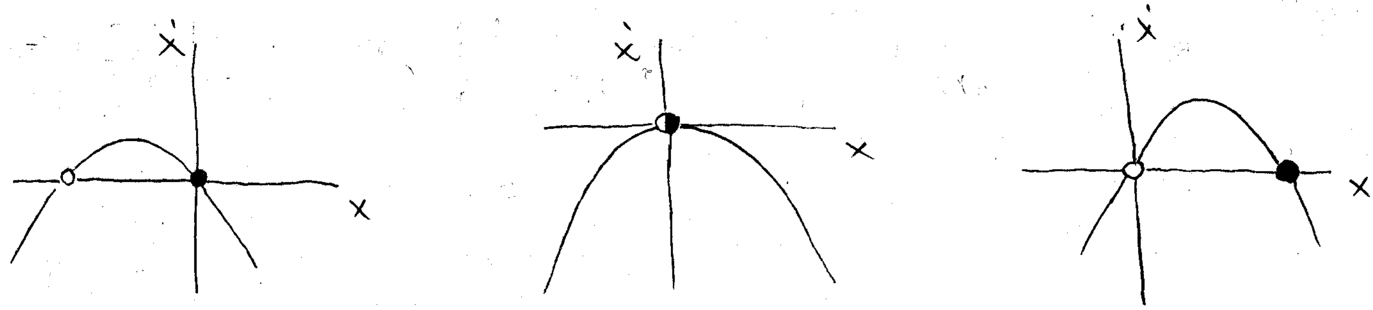
$$\Rightarrow \dot{x} = (r - r_c)\frac{\partial f}{\partial r}\Big|_{x^*, r_c} + (x - x^*)^2 \cdot \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x^*, r_c} + \dots$$

Rescale:

$$\left.\begin{aligned} r \to r' &= (r - r_c) \cdot \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\frac{\partial f}{\partial r} \\ x \to x' &= (x - x^*) \cdot \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \end{aligned}\right\} \Rightarrow \boxed{\dot{x}' = r' + x'^2}$$

Transcritical Bifurcation

Often one of the fixed points is parameter-independent, say $x^*(r) = 0$:



$r < r_c$ $\qquad$ $r = r_c$ $\qquad$ $r > r_c$

$$\left.\frac{\partial f}{\partial r}\right|_{x^*, r_c} = \left.\frac{\partial f}{\partial x}\right|_{x^*, r_c} = 0$$

$$\dot{x} = f(x,r) = (r - r_c)(x - x^*)\left.\frac{\partial^2 f}{\partial x \partial r}\right|_{x^*, r_c} + \frac{1}{2}(x - x^*)\left.\frac{\partial^2 f}{\partial x^2}\right|_{x^*, r_c} + \dots$$

For instance, for Logistic growth:

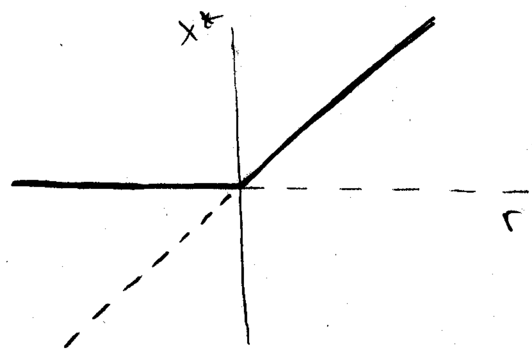
$$\dot{N} = rN\left(1 - \frac{1}{K}N\right) = rN - \frac{r}{K}N^2$$

Zero population ($N^* = 0$) is a fixed point for any growth rate r!

Normal form:

rescale x, r to get

$$\dot{x} = rx - x^2 \quad \Rightarrow \quad \begin{cases} x^* = 0 \\ x^* = r \end{cases}$$



Exchange of stability: two fixed points collide