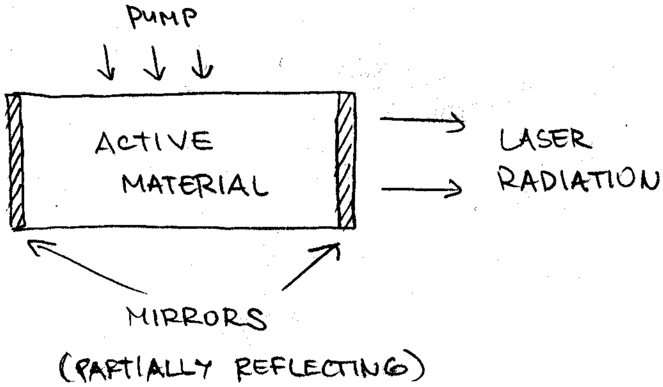
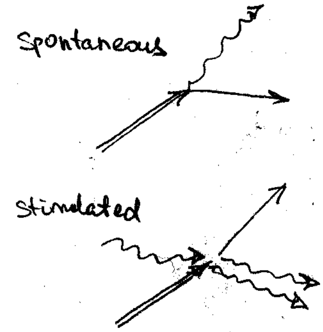
Laser Threshold





$$\dot{n} = GnN - kn$$ ← Number of coherent photons

Gain coefficient (G); # of excited atoms (N); rate of loss through mirror (k); # of photons (n)

Atoms emit photons in two ways:

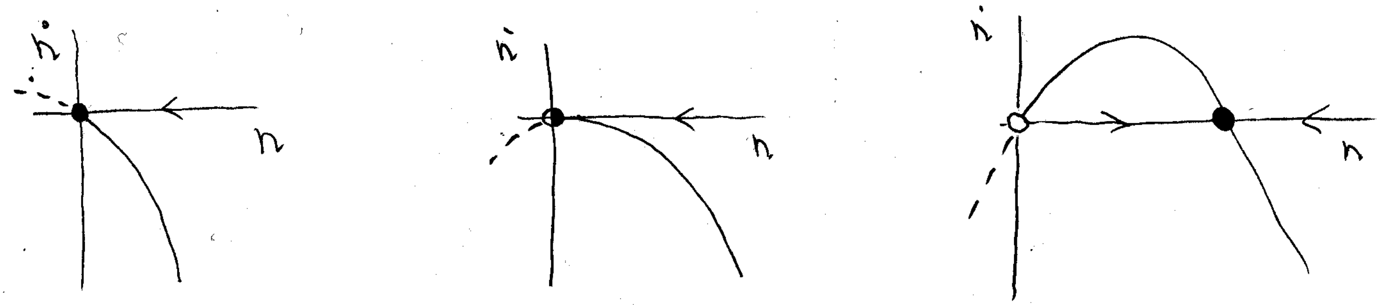
a) spontaneous emission – incoherent radiation (light bulb)

b) stimulated emission – coherent radiation (laser)

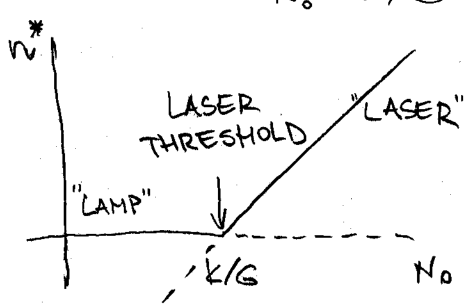
$$N = N_0 - \alpha n$$ ← Number of excited atoms

N_0: (pump) – (spontaneous emission); αn: stimulated emission

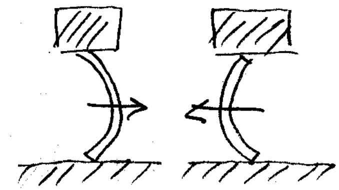
$$\dot{n} = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - (\alpha G)n^2$$



$N_0 < k/G$ $\qquad$ $N_0 = k/G$ $\qquad$ $N_0 > k/G$



Pitchfork Bifurcation



Bifurcation in the presence of symmetry:

$$f(x,r) = -f(-x,r)$$

1) $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x^*,r_c} = \left.\frac{\partial^4 f}{\partial x^4}\right|_{x^*,r_c} = \cdots = \left.\frac{\partial^{2n} f}{\partial x^{2n}}\right|_{x^*,r_c} = 0$

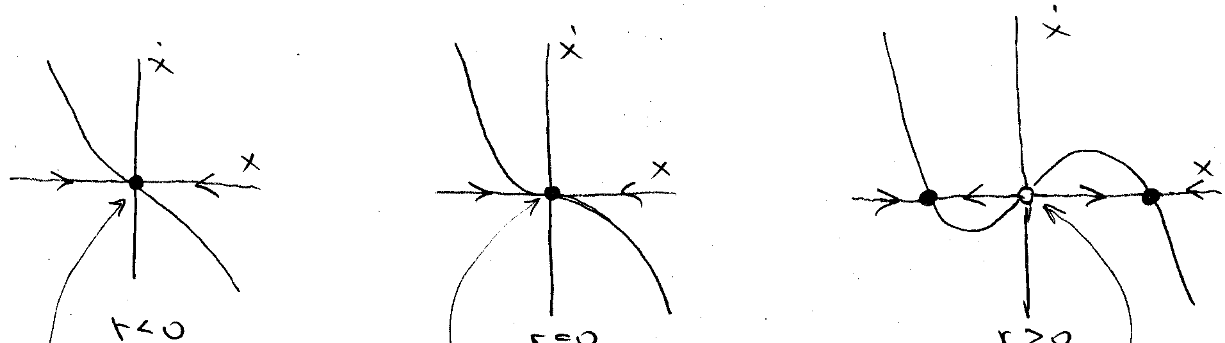
2) $x^* = 0$ is always a fixed point, so

$$\left.\frac{\partial f}{\partial r}\right|_{x^*,r_c} = 0$$

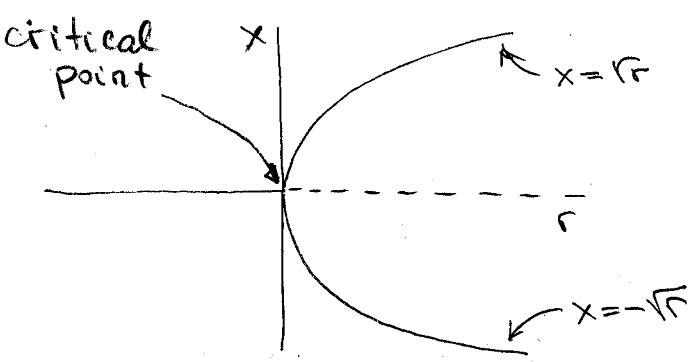
$$\dot{x} = f(x,r) = (r-r_c)(x-x^*)\left.\frac{\partial^2 f}{\partial x \partial r}\right|_{x^*,r_c} + \frac{1}{3!}(x-x^*)^3 \left.\frac{\partial^3 f}{\partial x^3}\right|_{x^*,r_c} + \cdots$$

Normal form:

$$\dot{x} = rx - x^3$$



exponential decay | algebraic decay (critical slowing down) | exponential growth



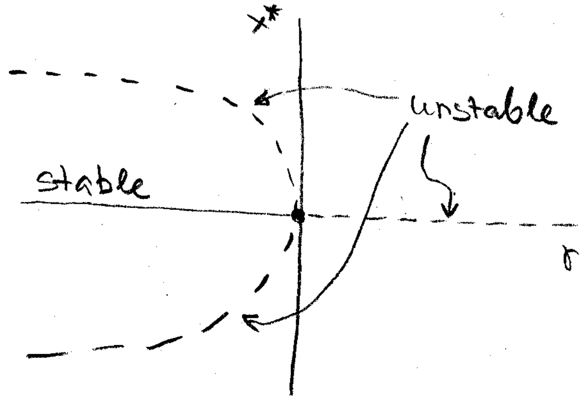
"supercritical" ($r>0$) pitchfork bifurcation

Also "forward" bifurcation (related to continuous or 2nd order phase transitions)

Supercritical Pitchfork Bifurcation

$\dot{x} = rx + x^3$

← now **destabilizing** term (important for finite x)



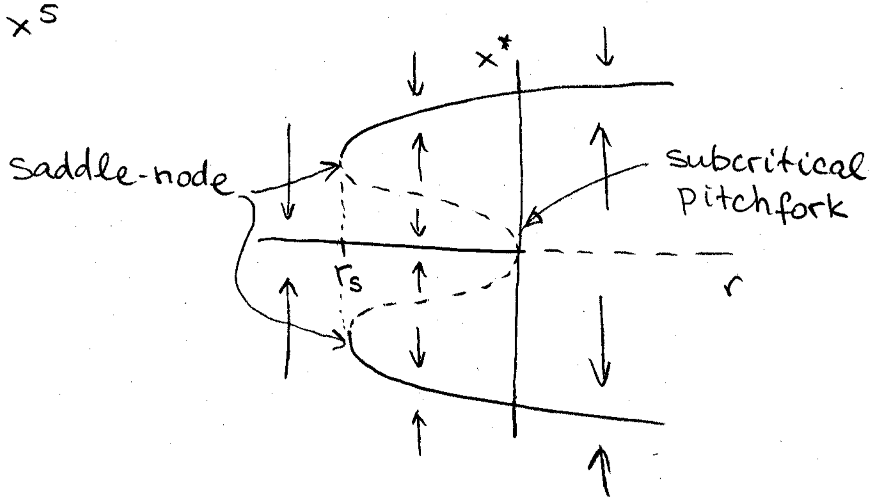
Also "inverted" or "backward" bifurc. (related to discontinuous or 1st order phase transitions)

"Subcritical" pitchfork (nontrivial solutions ∃ for $r<0$)

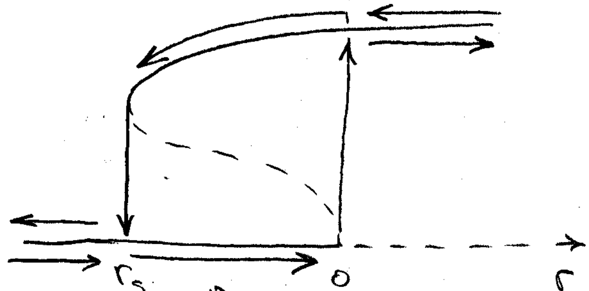
$f(x) \sim x^3$ for large x ⇒ solution blows up in finite time!

Under symmetry $x \to -x$ first stabilizing term is $\sim -x^5$:

$\dot{x} = rx + x^3 - x^5$



Hysteresis:



Bistability for $r_s < r < 0$