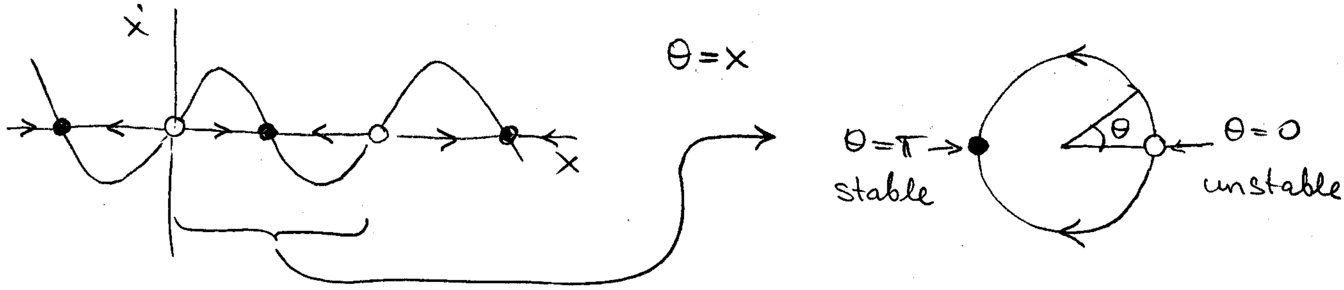
Flows on the circle

$$\dot{x} = \sin x \quad \leftarrow \text{periodic in } x$$



Vector field on a circle:

unique velocity for each point on the circle:

$$\dot{\theta} = f(\theta), \quad f(\theta + 2\pi) = f(\theta)$$

Remember: Oscillatory solutions possible!

Uniform oscillator

$$\dot{\theta} = \omega$$

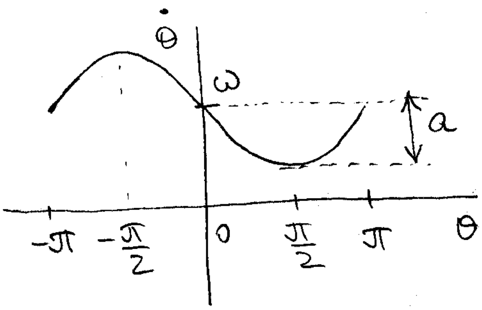
$$\theta(t) = \omega t + \theta_0 \quad - \text{ periodic}, \quad T = \frac{2\pi}{\omega}$$

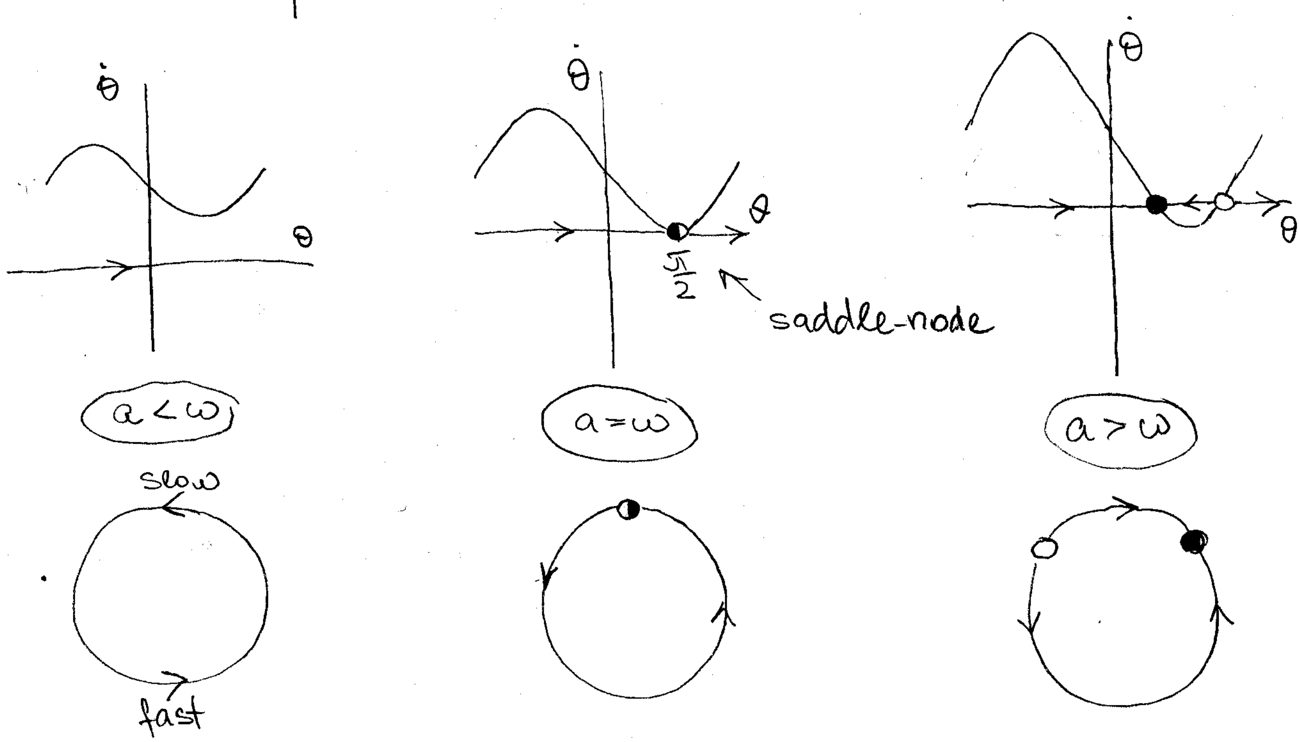
- $\theta(t)$ – phase of the system
- no amplitude (1-d system)

Non-uniform oscillator

$$\dot{\theta} = \omega - a\sin\theta$$

- phase locked loops (electronics)
- oscillating neurons
- fireflies
- Josephson junctions
- overdamped pendulum with torque





Linear stability:

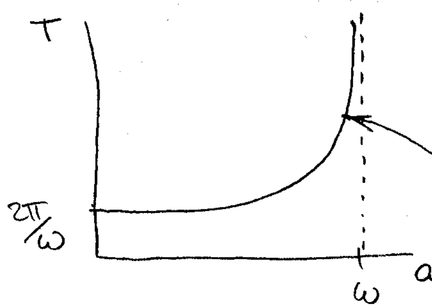
$$\sin\theta^* = \frac{\omega}{a}$$

$$f'(\theta^*) = -a\cos\theta^* = \mp a\sqrt{1-(\omega/a)^2}$$

$\cos\theta^* > 0$ – stable, $\cos\theta^* < 0$ – unstable

Oscillation period

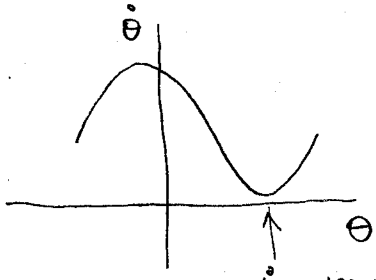
$$T = \int dt = \int \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} \underset{u = \tan\theta/2}{=} \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$



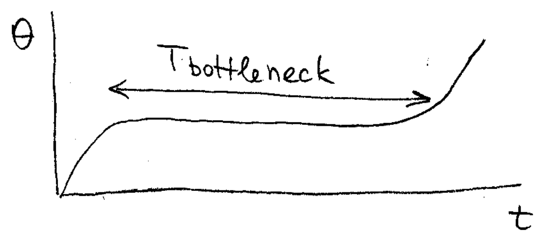
$$\sqrt{\omega^2 - a^2} = \sqrt{\omega + a}\,\sqrt{\omega - a} \approx \sqrt{2\omega}\,\sqrt{\omega - a}$$

$$\Rightarrow T \approx \pi\sqrt{\frac{2}{\omega}}\,\frac{1}{\sqrt{\omega - a}}\,, \quad a \to \omega^-$$

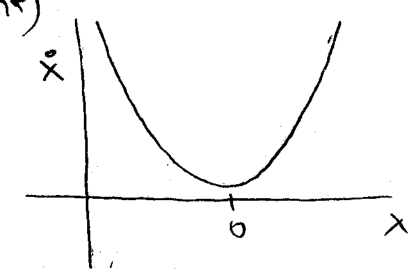
Ghosts & Bottlenecks:



$\Rightarrow$



$\dot\theta$ – very small (ghost of a fixed point)



saddle-node $\to$ locally parabolic

$\Rightarrow$ use normal form $\quad \dot{x} = r + x^2$

$$T_{\text{bottleneck}} \approx \int_{-\infty}^{+\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}} \quad \leftarrow \text{square root scaling}$$

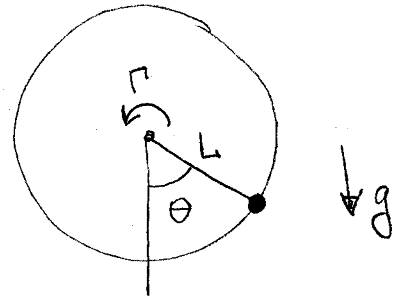
In our example: $\quad f(\theta) = \omega - a\sin(\theta) = \omega - a\left(1 - \frac{1}{2}\left(\theta - \frac{\pi}{2}\right)^2 + \dots\right)$, where $\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$

$$\approx \omega - a + \frac{1}{2} a \left(\theta - \frac{\pi}{2}\right)^2$$

Rescale: $\left.\begin{aligned} x &= (a/2)^{1/2}\varphi \\ r &= \omega - a \end{aligned}\right\} \Rightarrow (2/a)^{1/2}\,\dot{x} \approx r + x^2$

$$T \approx (2/a)^{1/2} \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = (2/a)^{1/2} \frac{\pi}{\sqrt{\omega - a}} \approx \pi\sqrt{\frac{2}{\omega}}\,\frac{1}{\sqrt{\omega - a}}$$

Overdamped Pendulum



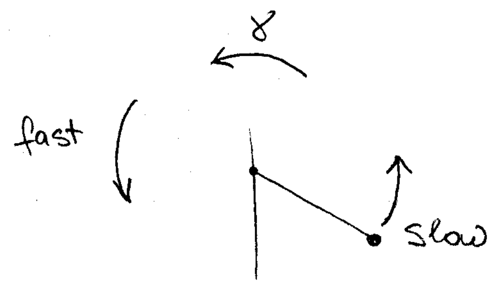
$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

$\uparrow$ strong damping $\Rightarrow$ drop

$$\left(\frac{b}{mgL}\right)\dot{\theta} = \left(\frac{\Gamma}{mgL}\right) - \sin\theta$$

$\left(\frac{b}{mgL}\right) \Downarrow T$: characteristic timescale

$\left(\frac{\Gamma}{mgL}\right) \Downarrow \gamma$: $\dfrac{\text{external torque}}{\text{max. gravit. torque}}$



$$\tau = t/T \; : \quad \frac{d\theta}{d\tau} = \theta' = \gamma - \sin\theta$$

Saddle-node bifurcation: $\gamma = 1$, $\theta^* = \frac{\pi}{2}$ (torques balance)

Fireflies

Fireflies influence each other $\Rightarrow$ synchronization

Periodic stimulus (flashing of other fireflies):

$$\dot{\Theta} = \Omega \,, \quad \Theta = 0 \Rightarrow \text{flash}$$

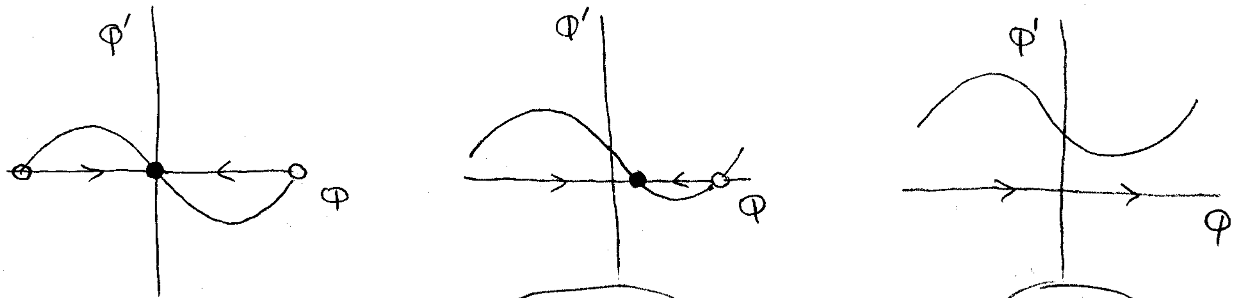
Firefly flashes:

$$\dot{\theta} = \omega + \underset{\substack{\uparrow \\ \text{resetting strength}}}{A}\sin(\Theta - \theta) \iff \begin{cases} \text{speed up, if behind stimulus} \\ \text{slow down, if ahead of stimulus} \end{cases}$$

Phase difference, $\varphi = \Theta - \theta$:

$$\dot{\varphi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\varphi \underset{\tau = At,\; \mu = \frac{\Omega - \omega}{A}}{\Longrightarrow} \boxed{\varphi' = \mu - \sin\varphi}$$

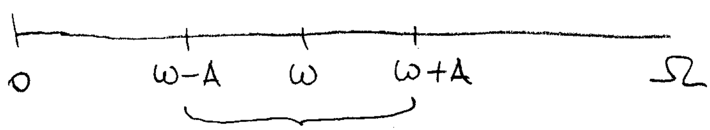
(Ermentrout & Rinzel)



$\mu = 0$	$0 < \mu < 1$	$\mu > 1$
simultaneous flashes	phase-locked (constant phase offset)	phase drift

$\underbrace{\text{simultaneous flashes} \quad \text{phase-locked}}_{\text{same period}}$ $\quad$ $\underbrace{\text{phase drift}}_{\text{different periods}}$

Same period — different periods



range of entrainment

Biological data:

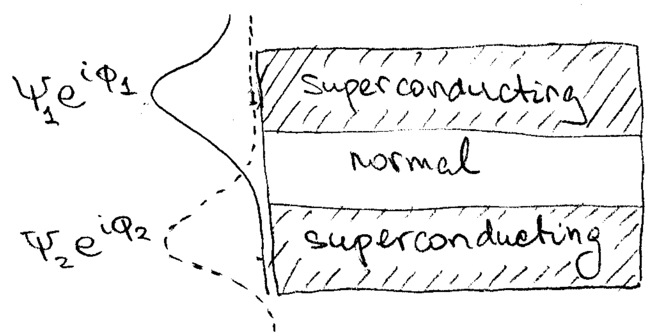
species with <u>fixed</u> ω — good agreement

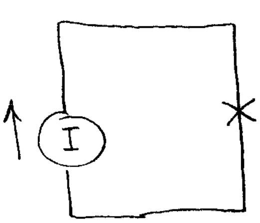
species with <u>variable</u> ω — poor agreement (expected)

<u>Superconducting Josephson junction</u>

<u>Ground state of SC:</u>

Coherent "Cooper pairs" of electrons





Phase: $\varphi = \varphi_1 - \varphi_2$

Current I

Voltage V

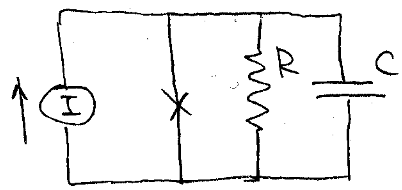
$I < I_c$: $\varphi = 0$, $V = 0$

$I \geq I_c$: $I = I_c \sin\varphi$,

$$V = \frac{\hbar}{2e}\dot{\varphi}$$

Real Josephson junction:

total current = Josephson current
+ ordinary current
+ displacement curr.



$$C\dot{V} + \frac{V}{R} + I_c \sin\varphi = I \quad (*) \qquad V = \frac{\hbar}{2e}\dot{\varphi}$$

$$\Rightarrow \frac{\hbar C}{2e}\ddot{\varphi} + \frac{\hbar}{2eR}\dot{\varphi} + I_c \sin\varphi = I$$ ← same equation as for damped pendulum.

Dimensionless form:

Timescale: $t_0 = \frac{\hbar}{2eI_cR} \quad \Rightarrow \tau = t/t_0$

McCumber parameter: $\beta = \frac{2eI_cR^2C}{\hbar}$ (dimensionless capacitance)

$$\beta\varphi'' + \varphi' + \sin\varphi = \frac{I}{I_c}$$

For small β (typically $10^{-6} \lesssim \beta \lesssim 10^{6}$)

$$\varphi' = \frac{I}{I_c} - \sin\varphi$$

I-V curve:

$$\langle V\rangle = \frac{\hbar}{2e}\langle\dot{\varphi}\rangle = \frac{\hbar}{2e}\frac{1}{t_0}\langle\varphi'\rangle = I_cR\langle\varphi'\rangle$$

$$\langle\varphi'\rangle = \frac{1}{T}\int_0^T \frac{d\varphi}{d\tau}d\tau = \frac{1}{T}\int_0^{2\pi} d\varphi = \frac{2\pi}{T} = \sqrt{(I/I_c)^2 - 1}$$

↖ already calculated

$$\langle V\rangle = I_cR\sqrt{(I/I_c)^2 - 1}, \quad I > I_c$$
$$= 0, \quad I < I_c$$