Two-dimensional Systems (Flows)

The dynamics of 2-d systems is much richer than that of 1-d systems.

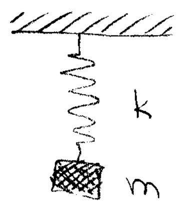
Study the simplest 2-d system (linear) to see what is new in 2-d.

Linear 2-d system:

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases} \iff \dot{\vec{x}} = A\vec{x} \;;\; \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix},\; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\vec{x} = 0$ – always a fixed point.

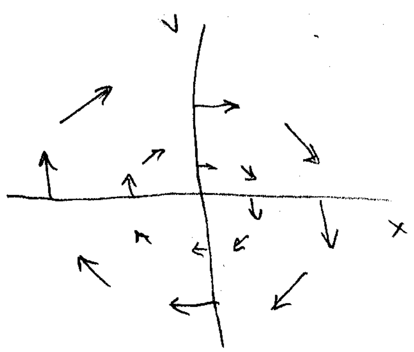
Example: (harmonic oscillator)



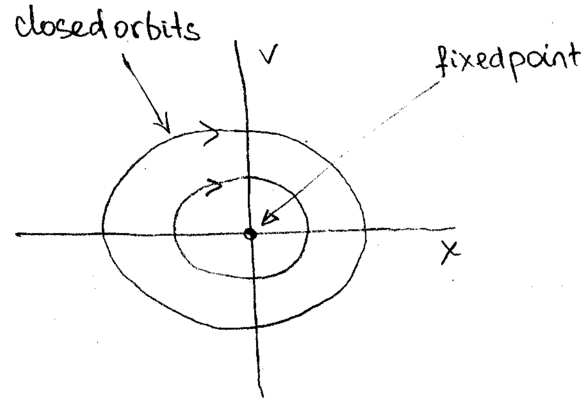
$$m\ddot{x} + kx = 0$$

$$\Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x = -\omega^2 x \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} x \\ v \end{pmatrix},\; A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

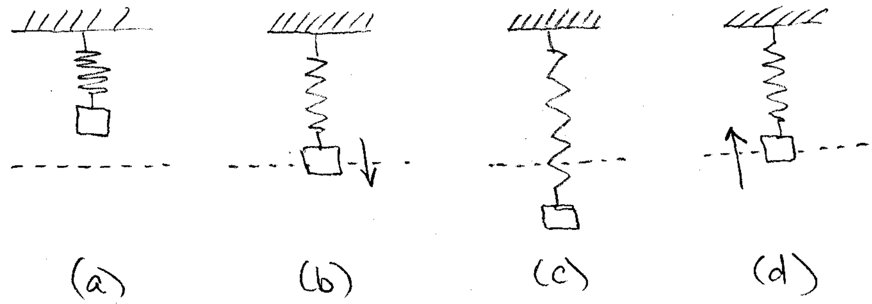
Vector field:



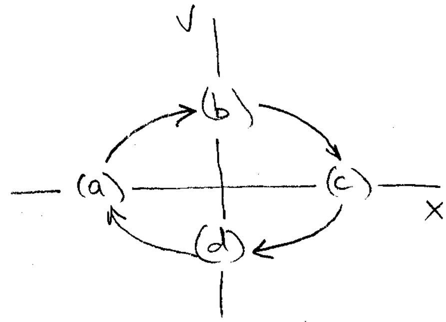
$\Rightarrow$



phase portrait



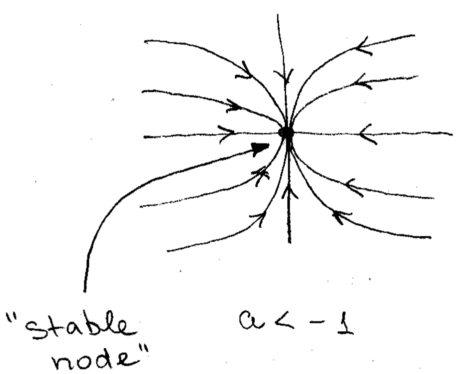
Closed orbit:



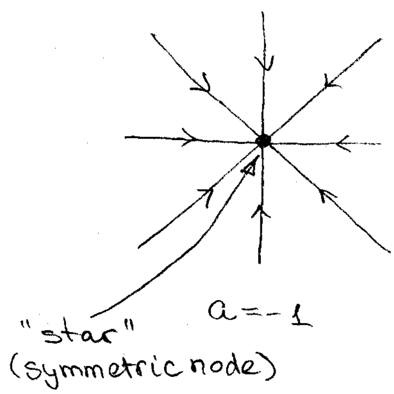
<u>Example:</u>

$$\dot{\vec{x}} = A\vec{x}, \quad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}, \quad -\infty < a < \infty$$

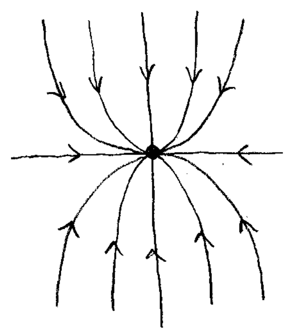
$$\begin{cases} \dot{x} = ax \\ \dot{y} = -y \end{cases} \Rightarrow \begin{cases} x = x_0 e^{at} & \leftarrow \text{depends on } a \overset{?}{\gtrless} 0 \\ y = y_0 e^{-t} & \leftarrow \text{decays exponentially} \end{cases}$$

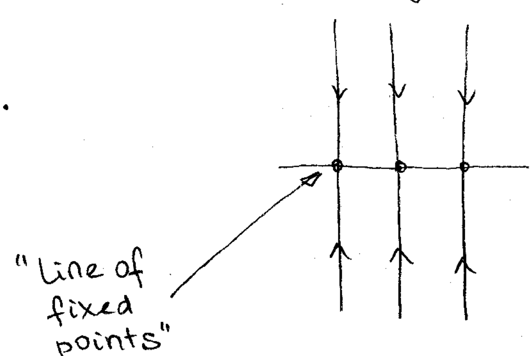


$a < -1$

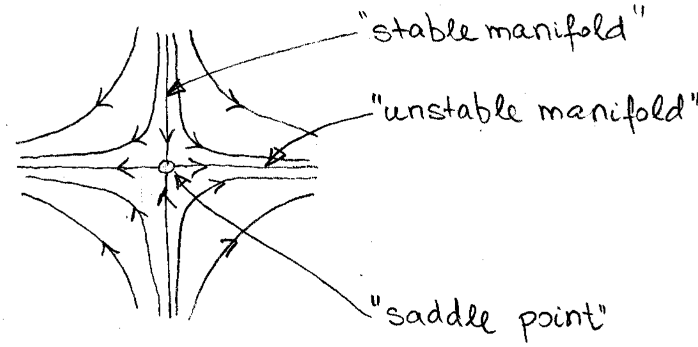


$a = -1$


$-1 < a < 0$



$a = 0$



$a > 0$

Types of Stability

- **Attracting fixed point:** $x(t) \to x^*$, $t \to \infty$ ($x(0)$ close to x^*)

  (Globally attracting: $x(t) \to x^*$, $\forall x(0)$)

- **Lyapunov stable fixed point:**

  $x(0)$ close to $x^* \Rightarrow x(t)$ close to x^*, $\forall t$

- **Neutrally stable fixed point:**

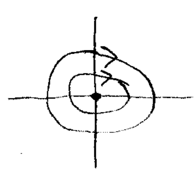
  Lyapunov stable, but not attracting

- **(Asymptotically) stable:**

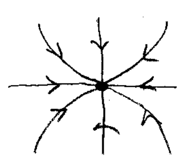
  Liapunov stable and attracting

- **Unstable:** Neither attracting nor Lyapunov stable

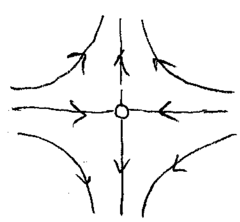
Examples:



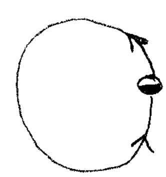
neutrally stable



asymptotically stable



unstable



Attractive, but not Lyapunov stable

Classification of Linear Systems

x, y - special coordinates (simple exponential growth / decay)

$\vec{x}(t) = e^{\lambda t}\vec{v} \quad \Rightarrow \quad \lambda e^{\lambda t}\vec{v} = A e^{\lambda t}\vec{v}$

$$\Rightarrow \boxed{A\vec{v} = \lambda\vec{v}}$$

eigenvalue (λ), eigenvector ($\vec{v}$) of A

Characteristic equation:

$$\det(A - \lambda I) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - \underbrace{(a+d)}_{\tau = Tr(A)}\lambda + \underbrace{(ad-bc)}_{\Delta = \det(A)} = 0$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \quad \Rightarrow \quad \begin{cases} \lambda_1 + \lambda_2 = \tau \\ \lambda_1 \cdot \lambda_2 = \Delta \end{cases}$$

More generally:

1) $\sum_n \lambda_n = Tr(A)$

2) $\prod_n \lambda_n = \det(A)$

Eigenvalues: $(A - \lambda_i I)\vec{v}_i = 0 \rightarrow \vec{v}_i$ (up to a numerical factor)

General Solution:

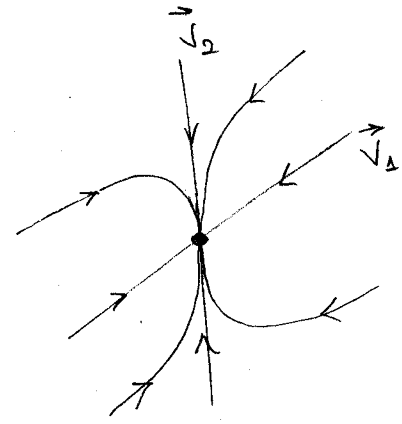
$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$ (assuming $\lambda_1 \neq \lambda_2$)

$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

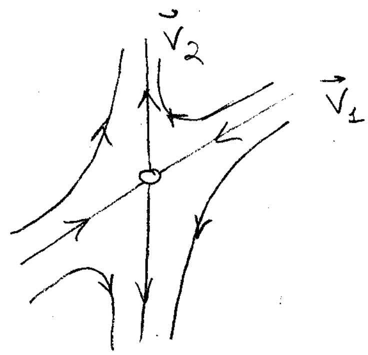
More generally: $\vec{x}(t) = \sum_n c_n e^{\lambda_n t}\vec{v}_n$

1) $\lambda_1 \neq \lambda_2$ – real

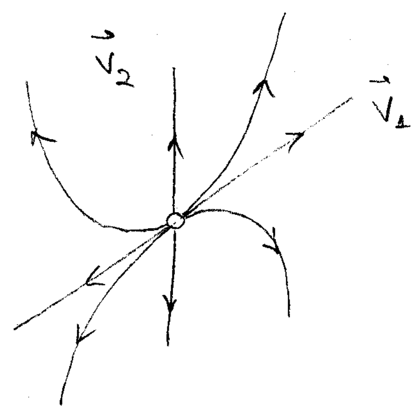
a) $\lambda_1 < \lambda_2 < 0$ (stable node)



b) $\lambda_1 < 0 < \lambda_2$ (saddle)

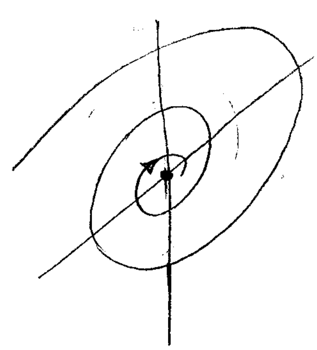


c) $0 < \lambda_1 < \lambda_2$ (unstable node)

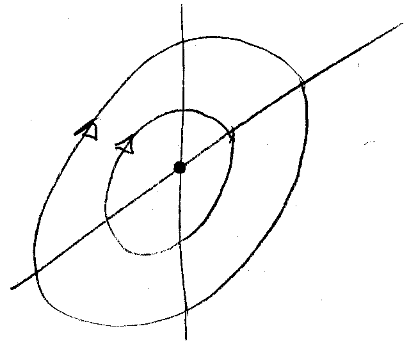


2) $\lambda_{1,2} = a \pm i\omega$ – complex

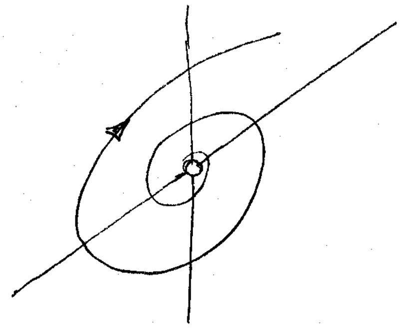
a) $a < 0$ (stable spiral)



b) $a=0$ (center)

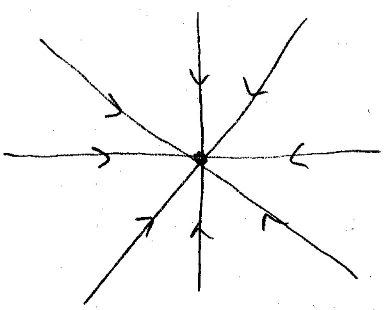


c) $a>0$ (unstable spiral)

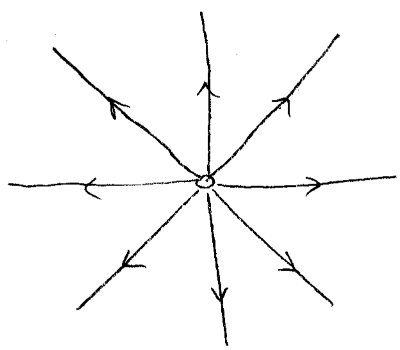


3) $\lambda_1=\lambda_2$ - real

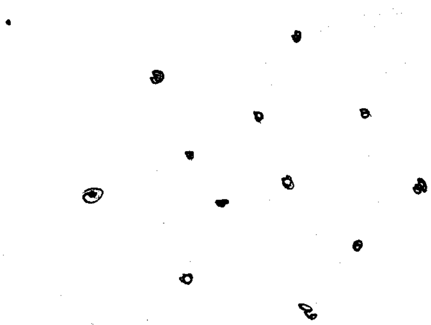
a) $A=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ => every vector is eigenvector



stable star node ($\lambda<0$)



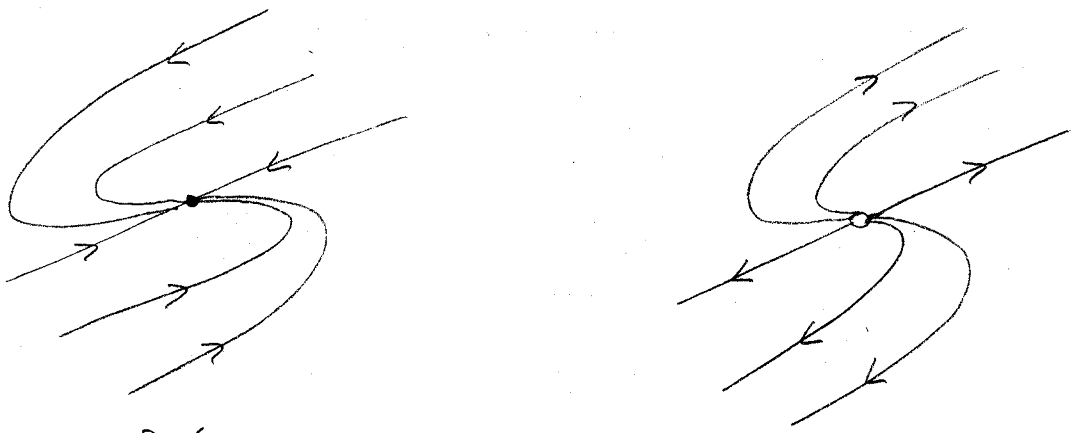
unstable star node ($\lambda>0$)



← every point is a fixed point ($\lambda=0$)

b) $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad b \neq 0$

Only <u>one</u> eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exists!



$\lambda < 0$ $\lambda > 0$

Classification diagram: