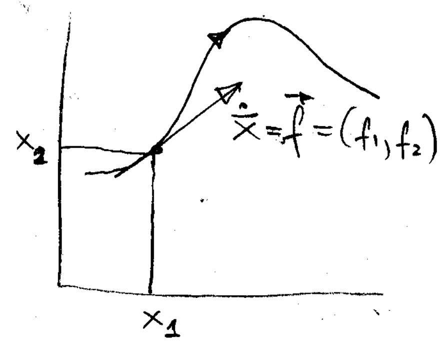
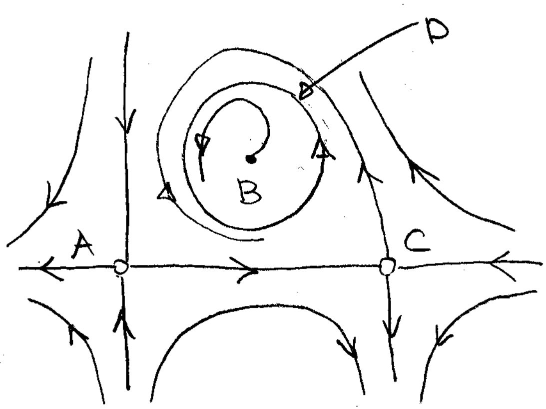
Phase plane Analysis

$$\underbrace{\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}}_{\text{2d nonlinear system}} \quad \text{or} \quad \dot{\vec{x}} = \vec{f}(\vec{x})$$



Goal:

Determine qualitative behavior based on the location and stability of fixed points:



- fixed points: A, B, C
- closed orbit: D
- local flows: A, C - similar; B - different
- stability: now of fixed points
  later of closed orbits (limit cycles)

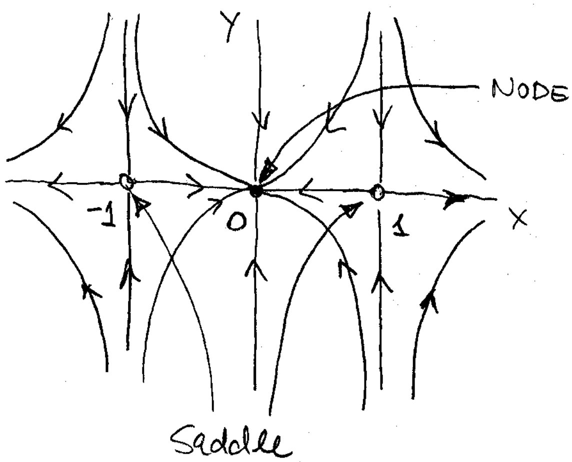
(0,0): $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ – stable node $\vec{v}_1 = (1,0)$

$\vec{v}_2 = (0,1)$

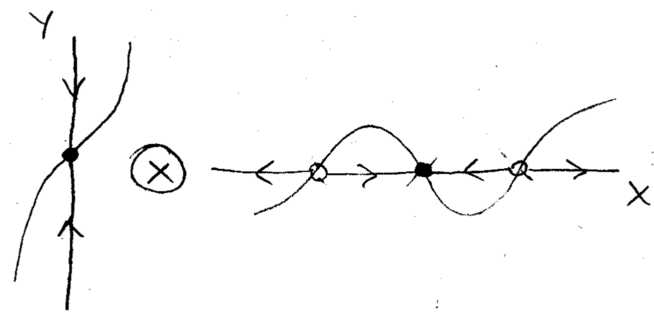
(±1,0): $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ – unstable node (saddle)

$\vec{v}_1 = (1,0)$ – unstable

$\vec{v}_2 = (0,1)$ – stable



$\Leftrightarrow$



Example: $\begin{cases} \dot{x} = -y + ax(x^2+y^2) \\ \dot{y} = x + ay(x^2+y^2) \end{cases}$ (0,0) – fixed point

Linearization (incorrect prediction)

$$A = \begin{bmatrix} a(3x^2+y^2) & -1+2axy \\ 1+2axy & a(3y^2+x^2) \end{bmatrix}_{0,0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\tau = 0,\ \Delta = 1 \Rightarrow \lambda_{1,2} = \pm\frac{1}{2}\sqrt{-4} = \pm i \Rightarrow \boxed{\text{center}}$

Nonlinear analysis: $(x,y) \to (r,\theta)$

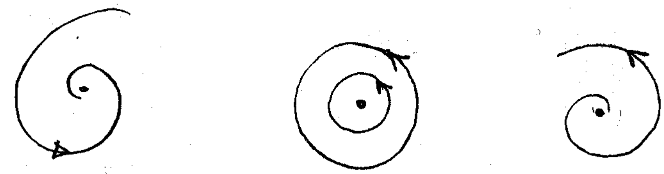
$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow x^2+y^2 = r^2$

$\frac{d}{dt}:\ 2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 2(-xy + ax^2r^2 + xy + ay^2r^2) = 2ar^4$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1 \quad \text{(prove - easy)}$$

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases} \Rightarrow \text{have 3 cases:}$$

$a<0$ $a=0$ $a>0$



spiral (stable) center spiral (unstable)

Centers - delicate: trajectory has to close perfectly after one cycle.

Hyperbolicity and Structural Stability

Hyperbolic fixed point: $Re(\lambda_i) \neq 0, \forall i$

Nonhyperbolic fixed point: $Re(\lambda_i) = 0$ for some i

$\boxed{\text{Very important notion:}}$

Structural stability: Hyperbolic - same type of f.p.
Nonhyperbolic - type of f.p. can change

Ergodicity: Hyperbolic - ergodic (mixing) dynamics
Nonhyperbolic - nonergodic (non-mixing)

Rabbits vs. Sheep

1. Each species would grow to carrying capacity in the absence of the other (logistic growth)
2. Both consume the same resource (grass)
3. Rabbits reproduce faster, sheep eat more

Lotka-Volterra equations for competition:

$$\begin{cases} \dot{x} = x(3-x-2y) & \leftarrow \text{rabbits} \\ \dot{y} = y(2-x-y) & \leftarrow \text{sheep} \end{cases}$$

fixed points:
a) $(0,0)$ – both species extinct
b) $(0,2)$ – sheep, no rabbits
c) $(3,0)$ – rabbits, no sheep
d) $(1,1)$ – sheep and rabbits

Jacobian:

$$A = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

$(x^*,y^*) = (0,0)$: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 3,\ \lambda_2 = 2$ (unstable node)

$v_1 = (1,0),\ v_2 = (0,1)$

$(x^*,y^*) = (0,2)$: $A = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$ (stable node)

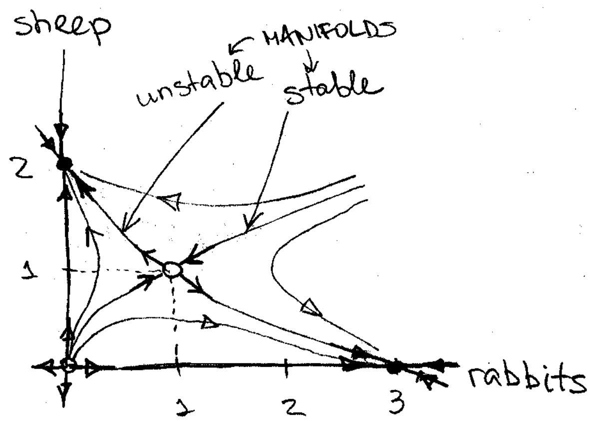
$v_1 = (1,-2),\ v_2 = (0,1)$

$(x^*,y^*) = (3,0)$: $A = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda_1 = -3,\ \lambda_2 = -1$ (stable node)

$v_1 = (1,0),\ v_2 = (3,-1)$

$(x^*,y^*) = (1,1)$: $A = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{2}$ (saddle point)

$v_{1,2} = (\sqrt{2} \mp 2, \sqrt{2} \mp 1)$



Competitive exclusion:

either sheep
or rabbits

$\boxed{\text{Basins of attraction}}$