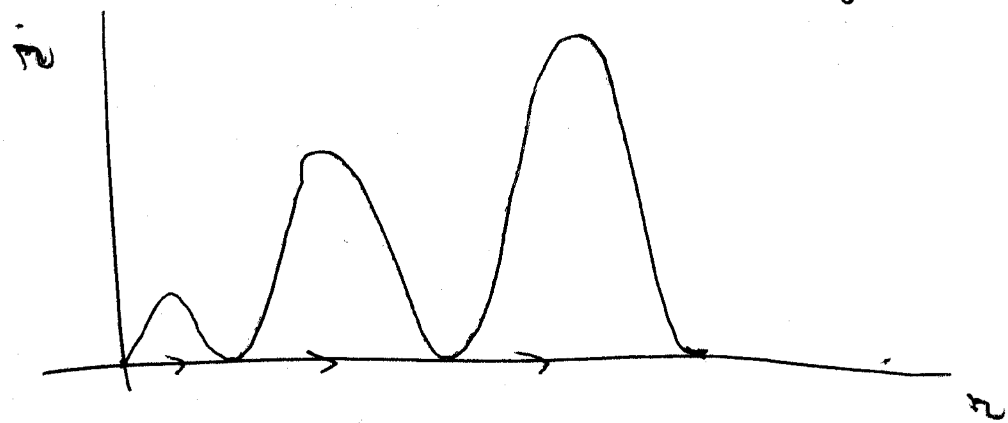
$$\begin{cases} \dot{r} = r(\mu - \sin r) \\ \dot{\theta} = 1 \end{cases}$$

i) $\mu > 1$ no cycles.

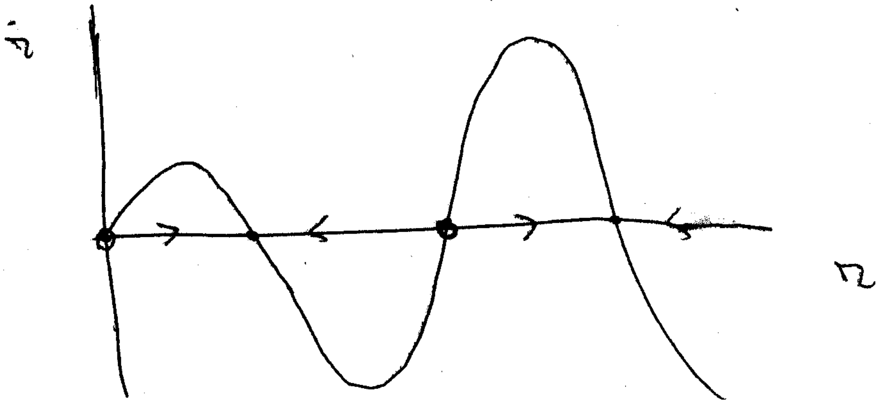
ii) $\mu = 1 \Rightarrow 1 - \sin r = 0 \Rightarrow \sin r = 1$

$\Rightarrow r = \frac{\pi}{2}(2k+1)$.



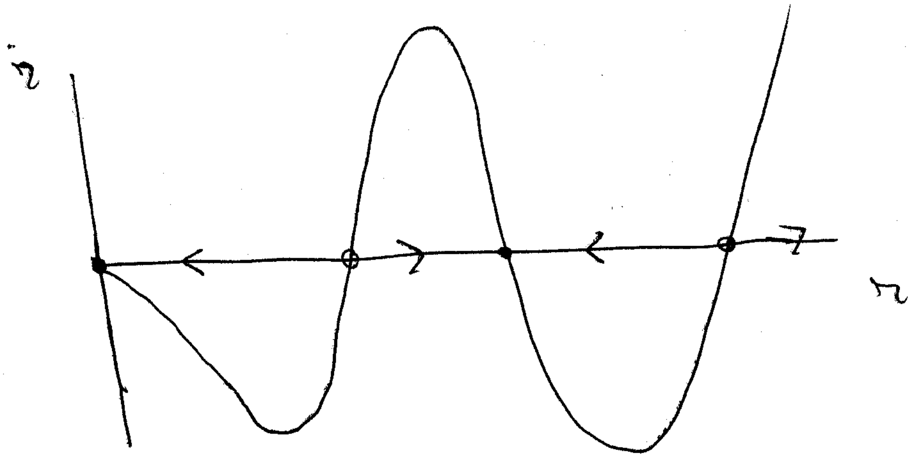
cycles are semistable

iii)



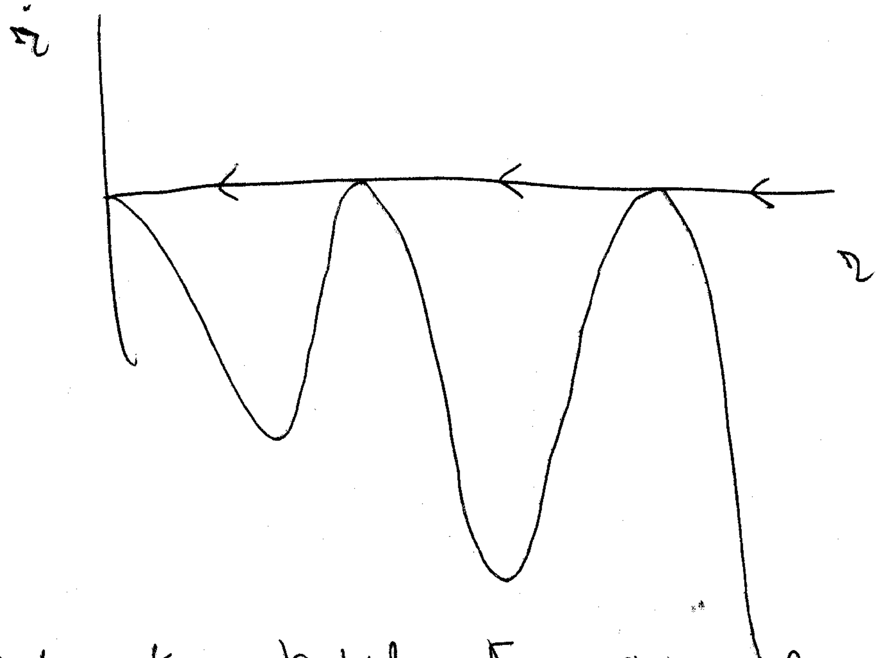
$0 < \mu < 1$

iv)



$-1 < \mu < 0$.

v)



$\mu < -1$

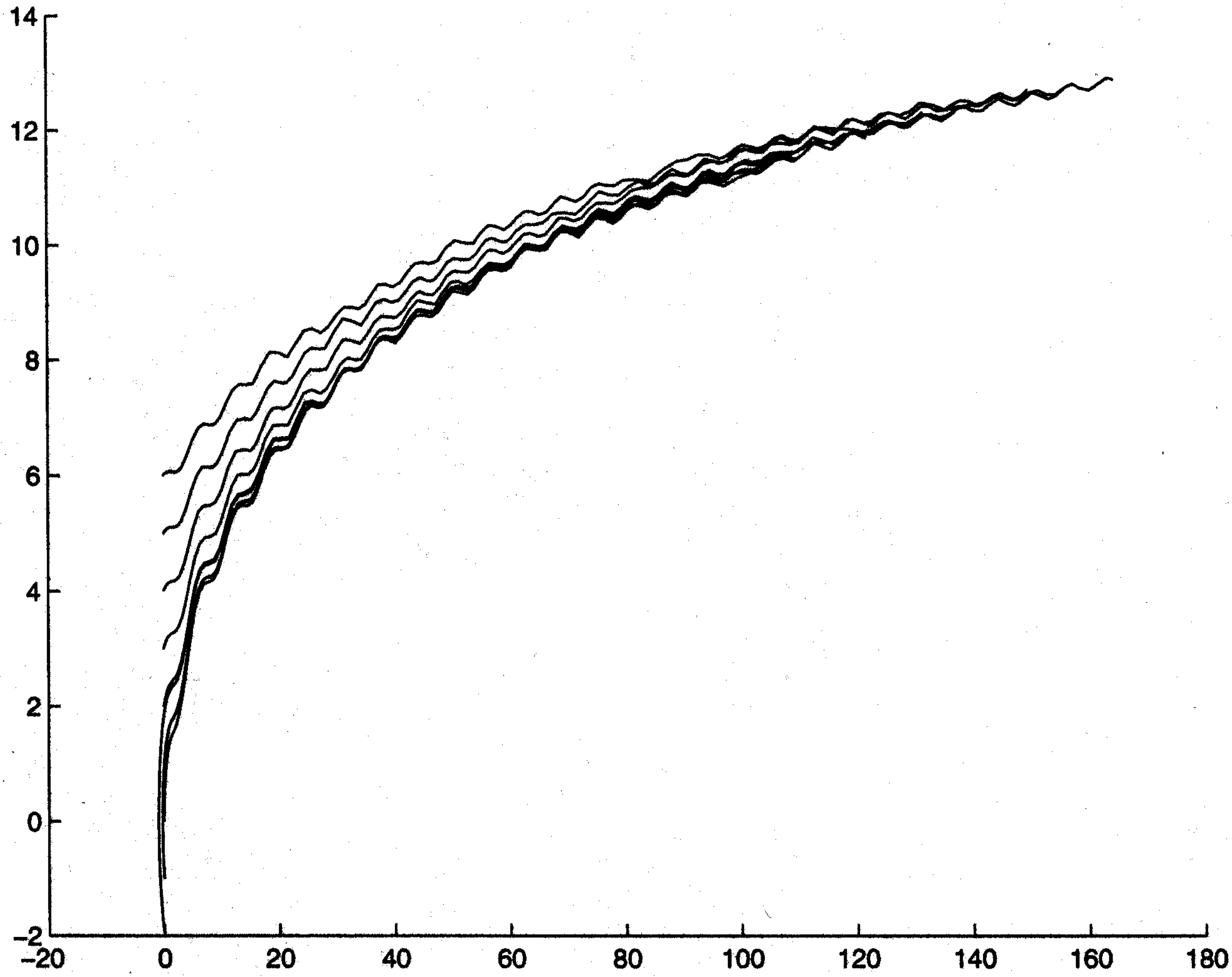
In summary, tangent bifurcations take place at $\mu = \pm 1$ and the fixed point at $r = 0$ undergoes a Hopf bifurcation at $\mu = 0$.

8.5.2

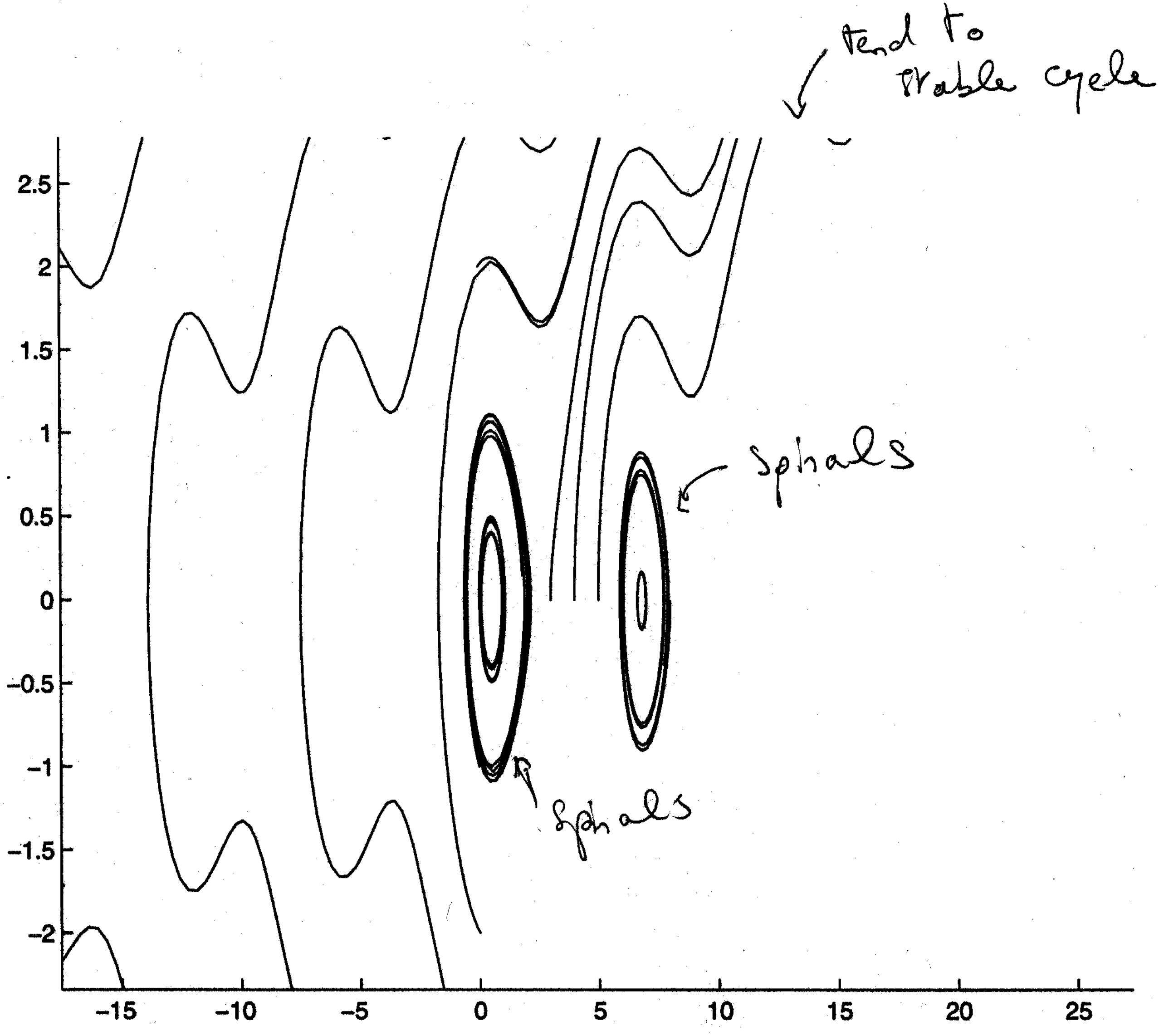
$$\begin{cases} x' = y \\ y' = I - \alpha y - \sin x \end{cases}$$

i) $I = 1.5, \quad \alpha = 0.01$

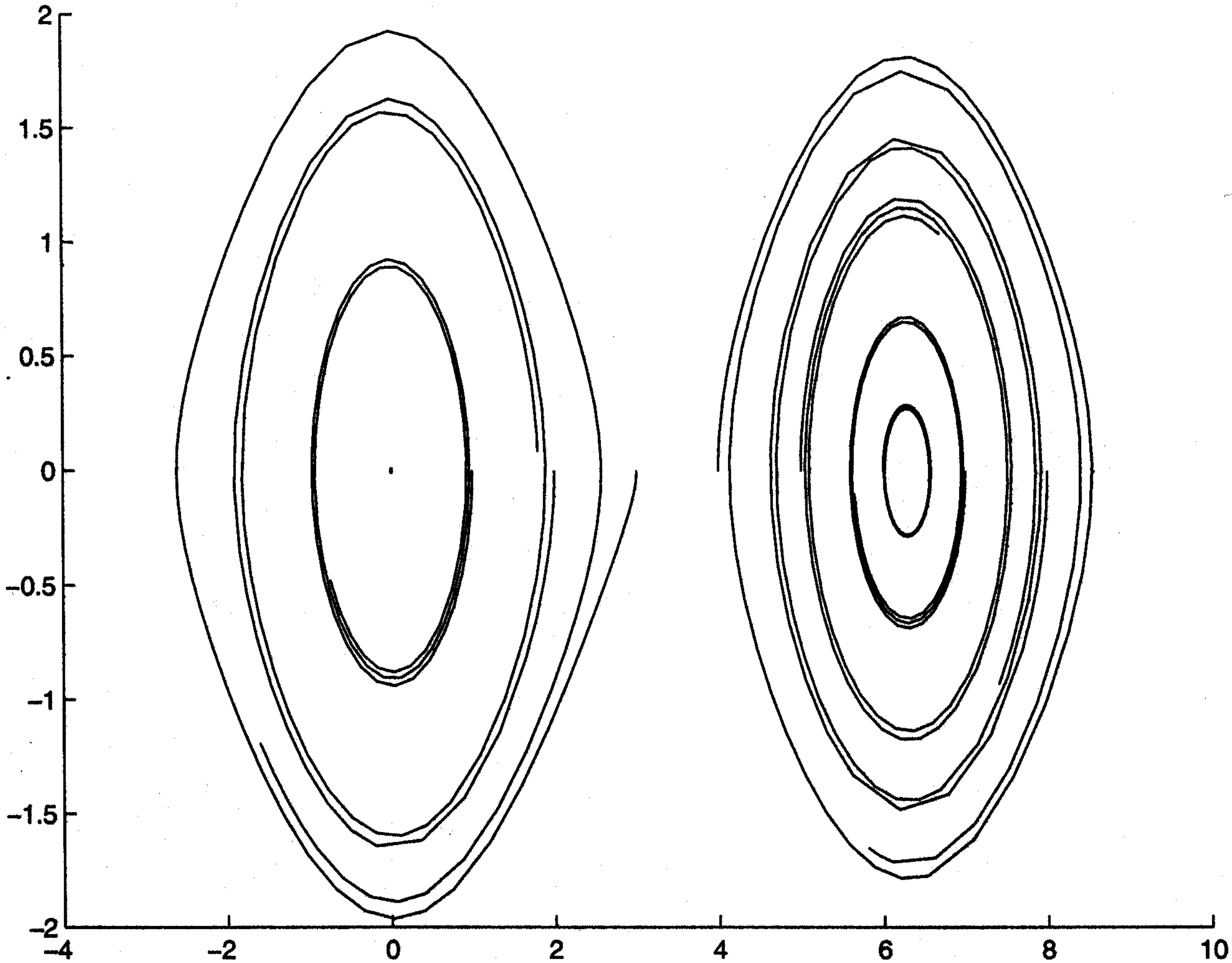
the system has a unique limit cycle



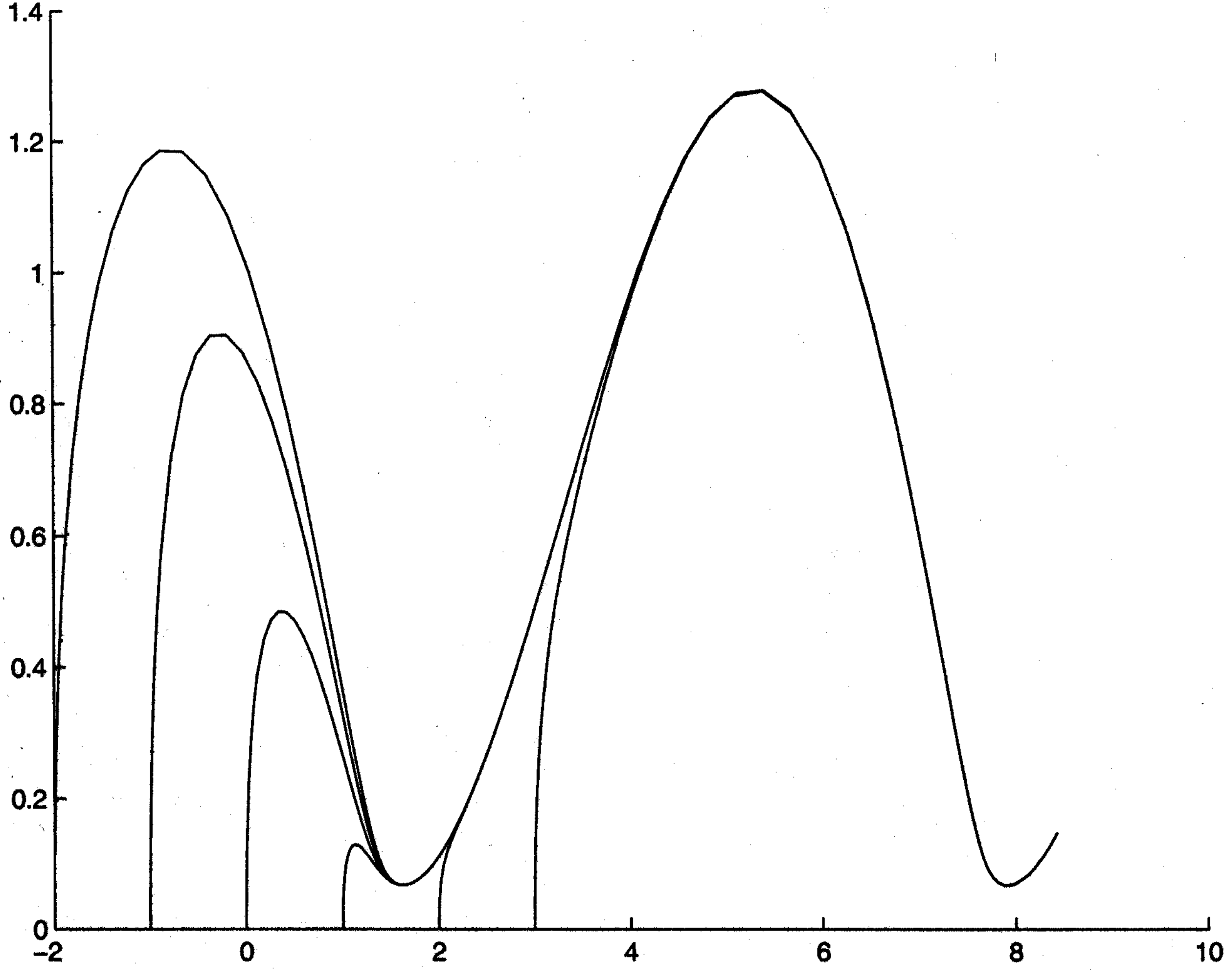
ii) As ~~I~~ I decreases ($I = 0.5$ in the figure below) the stable cycle and ~~sinks~~ sinks coexist



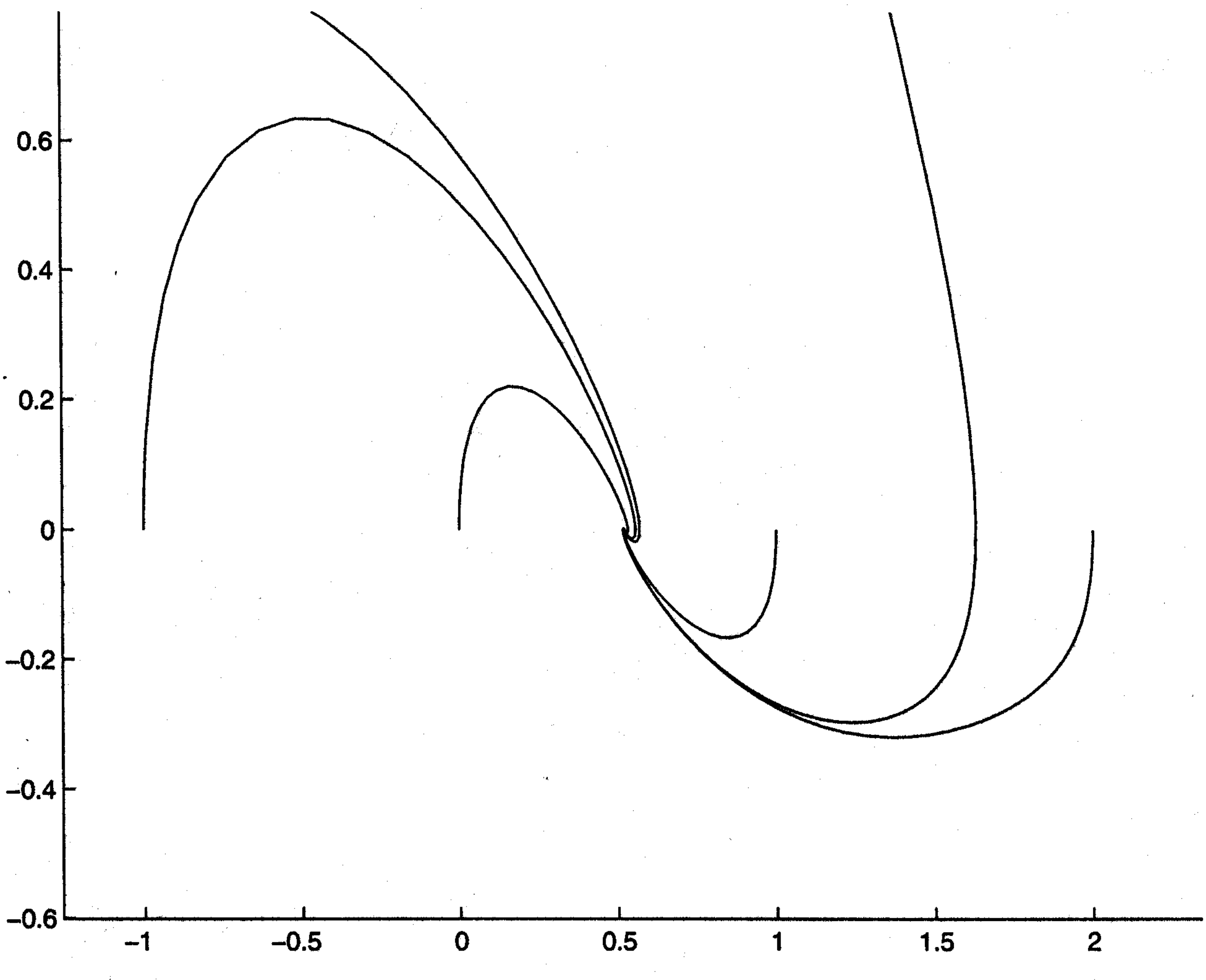
iii) As I goes below a critical value (in this case $I_c \approx 0.01$) only the stable fixed points survive.



iv) Here $\alpha = 1.5$, $I = 1.5$. All the trajectories still converge to a stable cycle.



v) As we lower I (I = 0.9 in the figure below), there is ~~never~~ coexistence between stable cycle and sinks. The cycle breaks and a series of sinks is created.



8.4.12

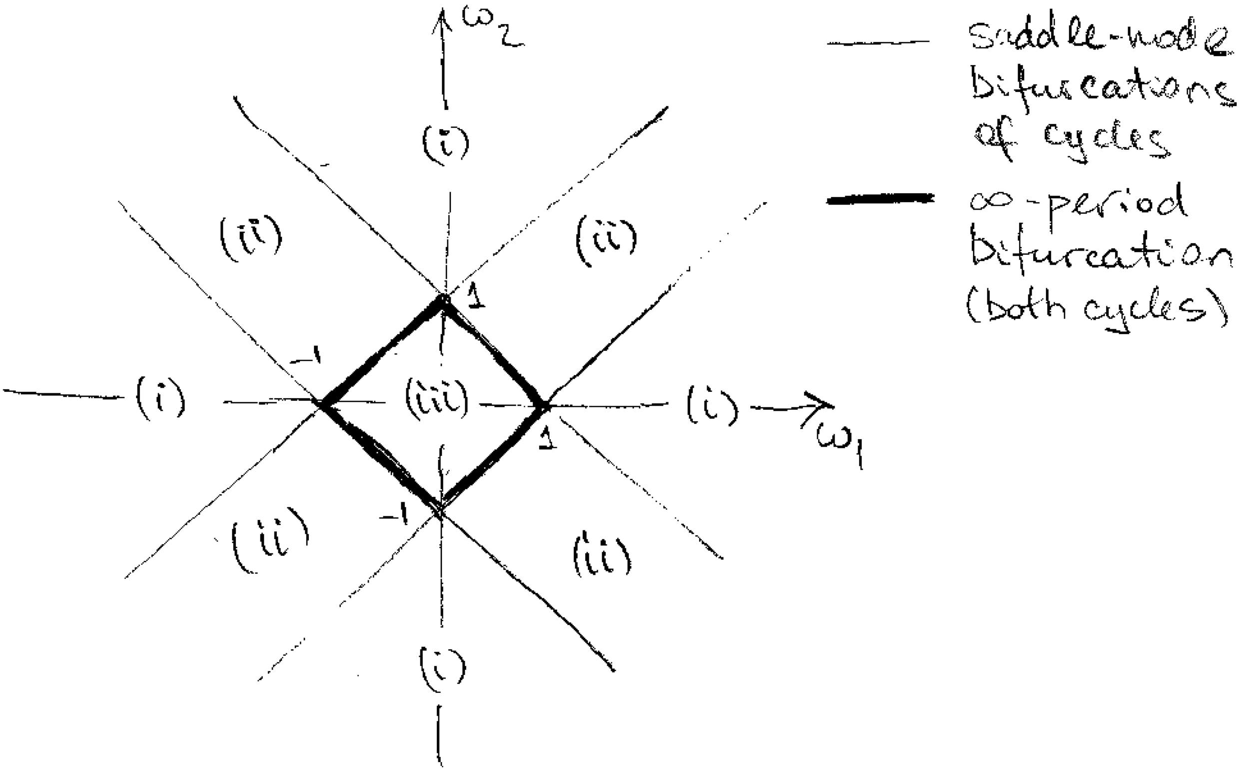
A combination of $\dot{x} = \lambda_u x$ and initial condition $x(0) = \mu$ gives
$$x(t) = \mu e^{\lambda_u t}$$

Using this solution we find
$$x(t) = 1 = \mu e^{\lambda_u t} \rightarrow t = -\frac{1}{\lambda_u} \ln \mu$$

8.6.1 Changing coordinates to $\varphi = \theta_1 - \theta_2$ and $\Psi = \theta_1 + \theta_2$ and introducing new parameters $\mu = \omega_1 - \omega_2$ and $\nu = \omega_1 + \omega_2$, we find
$$\begin{cases} \dot{\varphi} = \mu + \sin\varphi & (1) \\ \dot{\Psi} = \nu + \sin\Psi & (2) \end{cases}$$

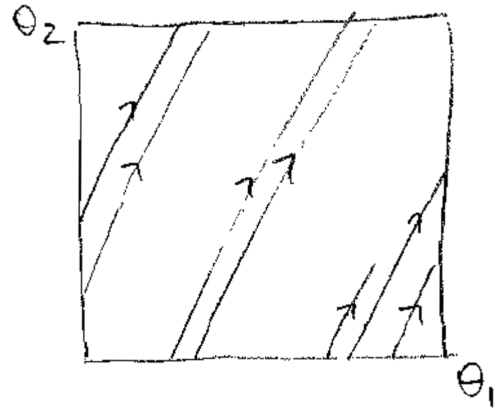
(1) has no f.p.'s for $|\mu| > 1$ and two for $|\mu| < 1$
(2) has no f.p's for $|\nu| > 1$ and two for $|\nu| < 1$

The stability diagram looks like this:



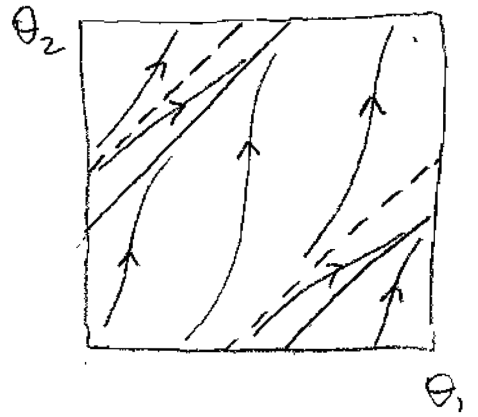
8.6.1 The qualitatively different phase portraits look like this:

case (i):



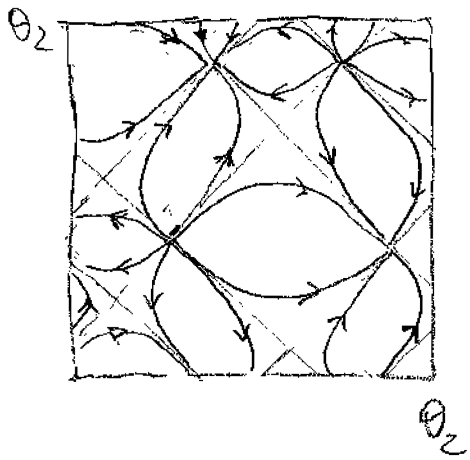
quasiperiodic motion

case (ii):



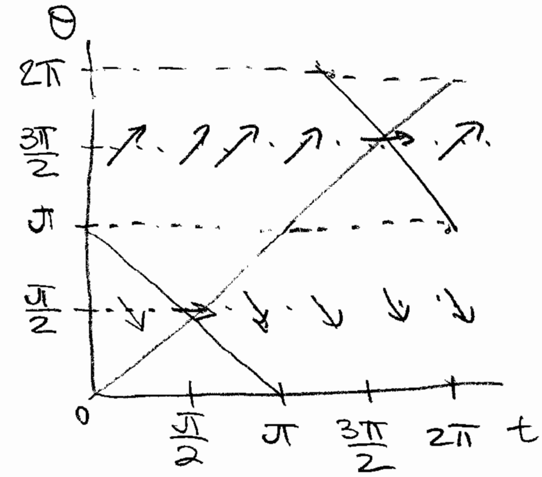
pair of limit cycles (one stable, one unstable)

case (iii):



fixed points (equal number of saddles and nodes)

8.7.5

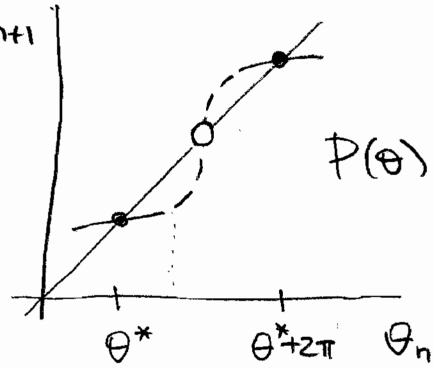


$\leftarrow \dot{\theta} \geq 0$ at $\theta = \frac{3\pi}{2}$

$\leftarrow \dot{\theta} \leq 0$ at $\theta = \frac{\pi}{2}$

method 1: Since $\dot{t} > 1$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is a trapping region which by Poincaré-Bendixson Thm. contains an attracting limit cycle.

Poincaré map (e.g., at $t = \frac{\pi}{2} + 2\pi n$) would have a corresponding fixed point at θ^* (and due to periodicity in θ) at $\theta^* + 2\pi$, etc. Continuity of the Poincaré map guarantees another intersection w/ diagonal, w/ slope > 1, corresponding to an unstable fixed point and hence unstable limit cycle.



method 2: Region $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ is a trapping region for the same system run in backward time and hence contains an unstable limit cycle (stable in backward time) by Poincaré-Bendixson theorem.

method 3: Vector field shows that $P(\frac{\pi}{2}) < \frac{\pi}{2}$ and $P(\frac{3\pi}{2}) > \frac{3\pi}{2}$ and hence there are at least two intersections w/ diagonal, one each for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.