10.4.6. Period 5 cycle (see 10.4.7. (b))

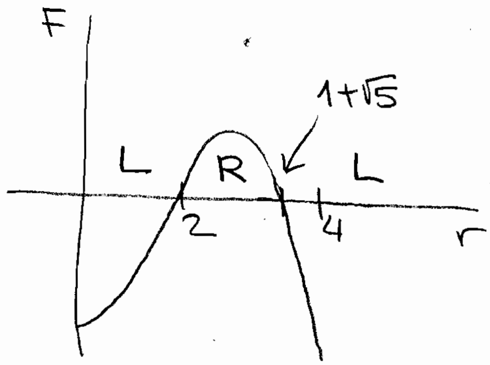
10.4.7. a)

$$x_0 = \frac{1}{2} \Rightarrow x_1 = r\frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{r}{4} > \frac{1+\sqrt{5}}{4} \approx 0.809 > 0.5 \Rightarrow R$$

$$x_2 = r\cdot\frac{r}{4}\left(1-\frac{r}{4}\right) < \frac{1}{2} \Leftrightarrow \frac{r^2}{4} - \frac{r^3}{16} - \frac{1}{2} < 0$$

Factoring the polynomial we find

$$F(r) \equiv \frac{r^2}{4} - \frac{r^3}{16} - \frac{1}{2} = \frac{1}{16}(2-r)(r-(1+\sqrt{5}))(r-(1-\sqrt{5})) < 0$$



$$\Downarrow$$

$x_2 < \frac{1}{2}$ for $r > 1+\sqrt{5}$

b)

$x_0 = 0.5$
$x_1 = 0.935 \rightarrow R$
$x_2 = 0.228 \rightarrow L$
$x_3 = 0.658 \rightarrow R$
$x_4 = 0.841 \rightarrow R$
$x_5 = 0.5$

10.5.2.

$$f(x) = 10x \bmod 1 \Rightarrow f'(x) = 10, \forall x$$

$$\Rightarrow \lambda = \langle \ln|f'(x_n)| \rangle_n = \langle \ln 10 \rangle_n = \ln 10.$$

$\boxed{10.6.3}$

(a) Logistic map

$x_0 = 0.5$
$x_1 = 0.90$
$x_2 = 0.306$
$x_3 = 0.7708$
$x_4 = 0.6409$
$x_5 = 0.8349$
$x_6 = 0.5$

$\Rightarrow$ RLRRR

6 cycle

(b) sine map.

$x_0 = 0.5$
$x_1 = 0.8811$
$x_2 = 0.3214$
$x_3 = 0.7461$
$x_4 = 0.6307$
$x_5 = 0.8079$
$x_6 = 0.5$

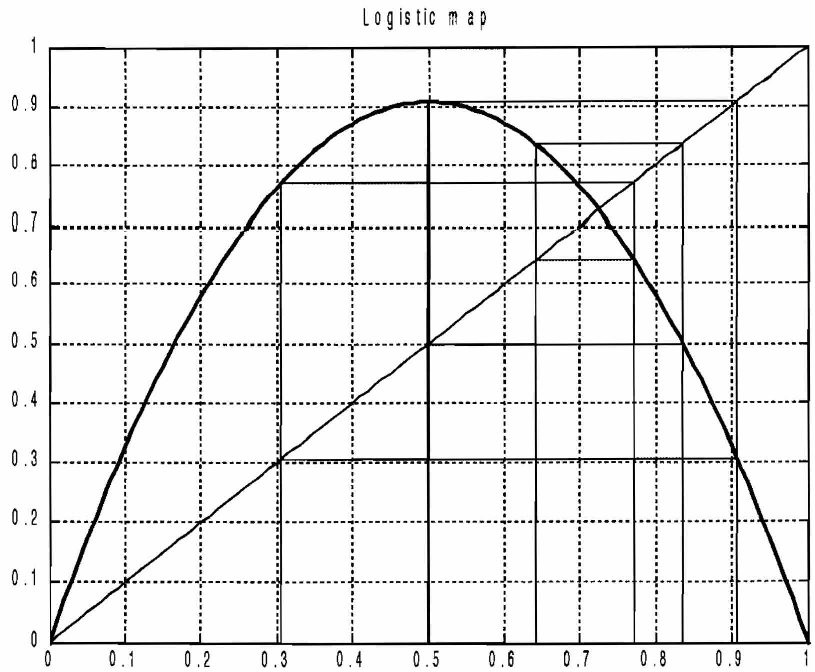
6 cycle

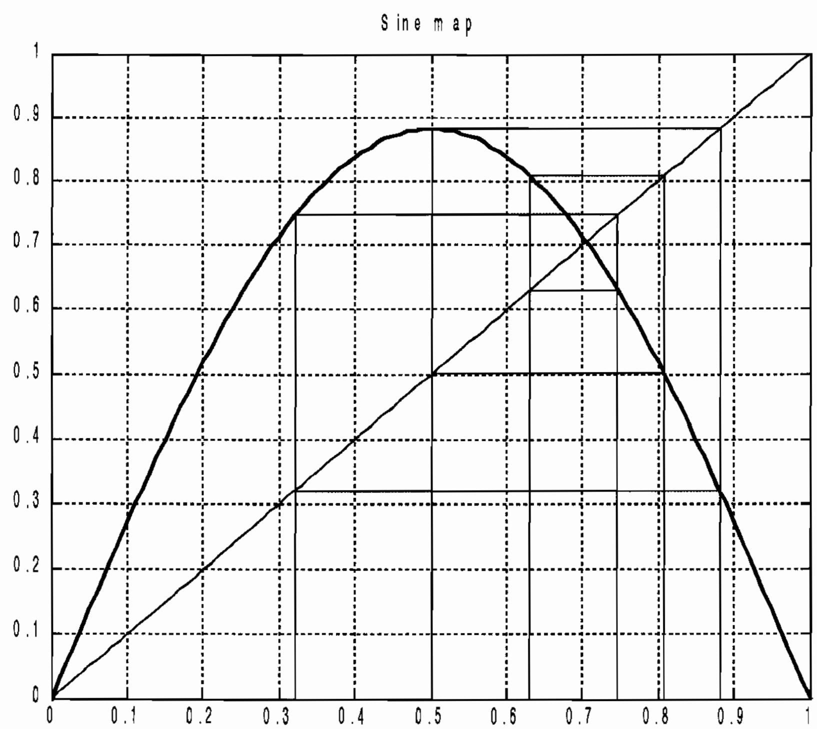
RLRRR

Patterns same but values differ

10.6.3

a)





b) For the logistic map, we get the iteration pattern: RLRRR

For the sine map, we get also: RLRRR

Problem 5

(a) $f(x) = \begin{cases} 2x & ; \ 0 < x < 1/2 \\ 1-x & ; \ 1/2 < x < 1 \end{cases}$

Note that $0 < f(x) < 1 \ \forall x \Rightarrow \int_0^1 \rho(x)\,dx = 1$

i.e; x_{n+1} stays in $(0,1)$ as long as $x_n \in (0,1)$

Now from Frobenius-Perron Eqn & the definition of invariant density, we have

$$\rho(x) = \int_0^1 \rho(y)\,\delta(x - f(y))\,dy$$

Assume $\rho(x) = \begin{cases} \rho_a & ; \ 0 < x < 1/2 \\ \rho_b & ; \ 1/2 < x < 1 \end{cases}$

$$\rho(x) = \int_0^{1/2} \rho(y)\,\delta(x-2y)\,dy + \int_{1/2}^{1} \rho(y)\,\delta(x-(1-y))\,dy.$$

$$= \rho_a \int_0^{1/2} \underbrace{\delta(x-2y)\,dy}_{y \leftrightarrow 2y} + \rho_b \int_{1/2}^{1} \underbrace{\delta(x-(1-y))\,dy}_{y \leftrightarrow 1-y}.$$

$$\rho(x) = \frac{\rho_a}{2}\int_0^1 \delta(x-y)\,dy + \rho_b \int_0^{1/2} \delta(x-y)\,dy$$

If $x \in (0, 1/2)$, second integral $= 1$, else 0.

$$\rho(x) = \frac{\rho_a}{2} + \rho_b \int_0^{1/2} \delta(x-y)\,dy.$$

For $x \in (0, 1/2)$; $\rho(x) = \rho_a = \frac{\rho_a}{2} + \rho_b \Rightarrow \rho_b = \frac{\rho_a}{2}$

For $x \in (1/2, 1)$; $\rho(x) = \rho_b = \frac{\rho_a}{2} + \cancel{\rho_b(0)}^{0} \Rightarrow \underline{\underline{\rho_b = \frac{\rho_a}{2}}}$

Now, since $\int_0^1 \rho(x)\, dx = 1$

$$\int_0^{1/2} \rho(x)\, dx + \int_{1/2}^1 \rho(x)\, dx = 1$$

$$\Rightarrow \int_0^{1/2} \rho_a\, dx + \int_{1/2}^1 \rho_b\, dx = 1$$

$$\Rightarrow \rho_a + \rho_b = 2$$

$$\Rightarrow \rho_a + \frac{\rho_a}{2} = 2 \Rightarrow \boxed{\rho_a = \frac{4}{3} \; ; \; \rho_b = \frac{2}{3}}$$

$\therefore$ Invariant density $\rho(x) = \begin{cases} \frac{4}{3} & ; \; x \in (0, 1/2) \\ \frac{2}{3} & ; \; x \in (1/2, 1) \end{cases}$

Lyapunov exponent

$$\lambda = \int_0^1 \rho(x) \ln |f'(x)|\, dx$$

$$= \int_0^{1/2} \rho_a \ln 2\, dx + \int_{1/2}^1 \cancel{\rho_b \ln 1}^{\,0}\, dx$$

$$= \frac{\rho_a}{2} \ln 2 = \frac{2}{3} \ln 2$$

$$\boxed{\lambda = \frac{2}{3} \ln 2}$$

(b) $f(x) = \begin{cases} rx & ; \ 0 < x < 1/2 \\ r(1-x) & ; \ 1/2 < x < 1 \end{cases}$

If $r < 2$, for $x \in (0, \frac{1}{2})$; $rx < 1$

for $x \in (\frac{1}{2}, 1)$; $1-x \in (0, \frac{1}{2})$ & $r(1-x) < 1$

$\therefore$ If $x_n \in (0,1)$; $x_{n+1} \in (0,1)$ $\Rightarrow$ $\int_0^1 \rho(x)\,dx = 1$

Frobenius - Perron gives

$$\rho(x) = \int_0^1 \rho(y)\, \delta(x - f(y))\, dy$$

Case(1): $r < 1$; Define $\rho(x) = \begin{cases} \rho_a ; & x \in (0, 1/2) \\ \rho_b ; & x \in (1/2, 1) \end{cases}$

$$\rho(x) = \int_0^{1/2} \rho_a\, \delta(x - ry)\, dy + \int_{1/2}^{1} \rho_b\, \delta(x - (1-y)r)\, dy$$

$$= (\rho_a + \rho_b) \int_0^{1/2} \delta(x - ry)\, dy$$

$$\rho(x) = \frac{(\rho_a + \rho_b)}{r} \int_0^{r/2} \delta(x - y)\, dy$$

* If $x \in (0, \frac{r}{2}) \subset (0, \frac{1}{2})$; $\rho(x) = \rho_a = \dfrac{\rho_a + \rho_b}{r}$

$\Rightarrow$ $\rho_a(r-1) = \rho_b$

If $x \in (\frac{r}{2}, \frac{1}{2})$; $\rho(x) = \rho_a = 0 \Rightarrow \rho_b = 0$!

$\therefore$ For $r < 1$; $\underline{\rho(x) = 0 \ \forall x \in (0,1)}$

This is because if we start at x_0 ; say $> \frac{1}{2}$; $1 - x_0 < \frac{1}{2}$;

$x_1 = r(1-x_0) < \frac{r}{2} < \frac{1}{2}$

$\Rightarrow x_2 = rx_1 = r^2(1-x_0) \in (0, \frac{1}{2})$

$x_n = r^n(1-x_0)$ $\Rightarrow$ $\lim_{n\to\infty} x_n = 0$ $\Rightarrow$ particle finally goes to 0

& spends <u>all</u> time there

Strictly, $\underline{\rho(x) = \delta(x) \text{ for } x \in [0,1)}$, so that $\int_0^1 \delta(x)\,dx = 1$

This makes sense because $|f'(x)|_{x=0}| = r < 1$ & $x = 0$ is a fixed point of the system & an attractor ~~at~~ that

$$\underline{\text{Lyapunov Exponent}}: \quad \lambda = \int_0^1 \rho(x) \ln |f'(x)|\,dx$$

$$\lambda = \int_0^1 \delta(x) \ln r\,dx \Rightarrow \underline{\lambda = \ln r}$$

For $r < 1$, $\lambda < 0 \Rightarrow$ stable system.

<u>Case (2)</u>: $r > 1$

$$\rho(x) = \frac{(\rho_a + \rho_b)}{r} \int_0^{r/2} \delta(x-y)\,dy$$

If $x \in (0, \frac{1}{2}) \subset (0, \frac{r}{2})$; $\rho(x) = \rho_a$ & $\int_0^{r/2} \delta(x-y)\,dy = 1$

$$\Rightarrow \quad \underline{\rho_a (r-1) = \rho_b}$$

If $x \in (\frac{1}{2}, \frac{r}{2})$; $\rho(x) = \rho_b$ & $\rho_b (r-1) = \rho_a$

Also $\int_0^1 \rho(x)\,dx = 1 \Rightarrow \rho_a + \rho_b = 2$

$$\Rightarrow \boxed{\rho_b = \frac{2}{r}} \; ; \; \boxed{\rho_a = 2 - \frac{2}{r}}$$

Lyapunov exponent; $\lambda = \int_0^1 \rho(x) \ln |f'(x)|\,dx =$

$$\Rightarrow \boxed{\lambda = \ln r} > 0 \Rightarrow \text{chaotic}$$