2.5.3

$\dot{x} = rx + x^3$, $r > 0$, $x = 0$ is a fixed point

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{rx + x^3} = \int_0^t dt$$

$$\Rightarrow \frac{1}{2r} \ln\left(\frac{x^2}{r + x^2}\right) = t - C_1 , \quad C_1 = -\frac{1}{2r} \ln\left(\frac{x_0^2}{r + x_0^2}\right).$$

$$\Rightarrow \frac{x^2}{r + x^2} = e^{2r(t - C_1)}$$

Now, when $x \to \pm\infty$ the left-hand side tends to 1,

so that

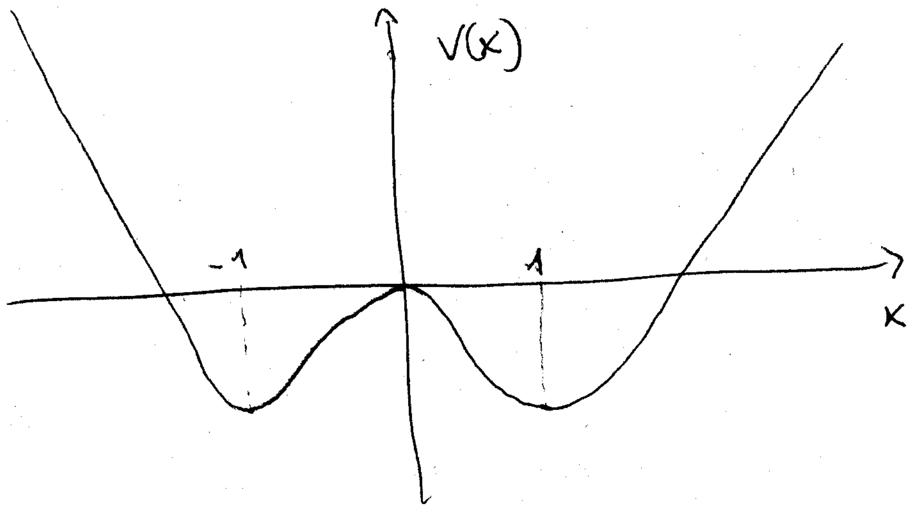
$$e^{2r(t - C_1)} \to 1 \Rightarrow 2r(t - C_1) = 0$$

$\Rightarrow t = C_1$ finite for any $x_0 \neq 0$

2.7.6

$$V(x) = \frac{x^4}{4} - \frac{1}{2}x^2 - rx$$

When $r = 0$, $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$ double-well potential

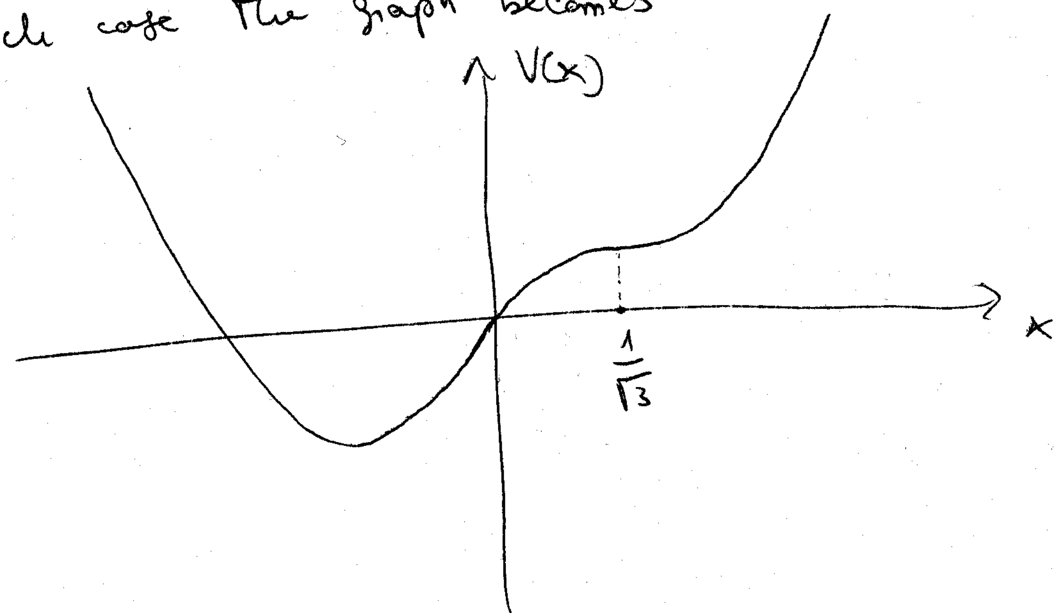


Now $V''(x) = -1 + 3x^2$ does not depend on r, and

$V''(x) > 0$ when $x > \frac{1}{\sqrt{3}}$ or $x < -\frac{1}{\sqrt{3}}$

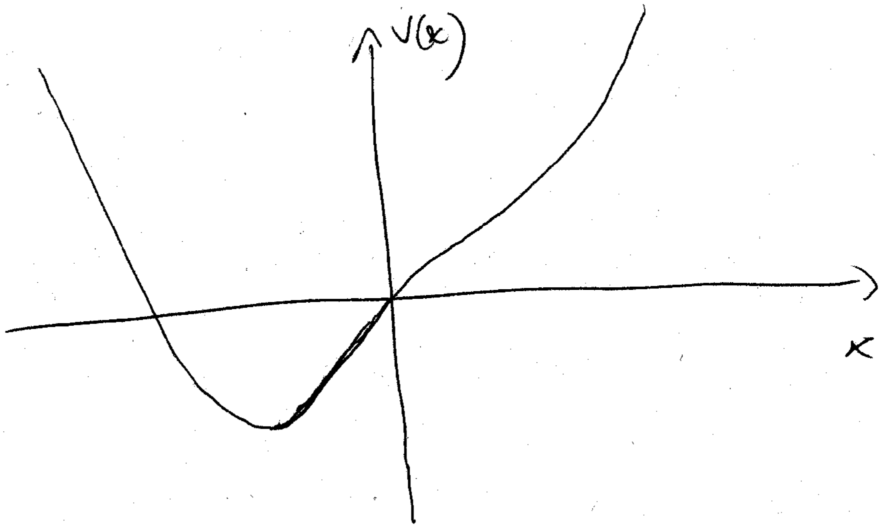
the two wells ~~are~~ exist until one of the two minima becomes a saddle point, that is $V'(x^*) = 0 = V''(x^*)$

$V'\left(\frac{1}{\sqrt{3}}\right) = -r - \frac{2}{3^{3/2}} = 0$ iff $r = -\frac{2}{3^{3/2}}$, in which case the graph becomes

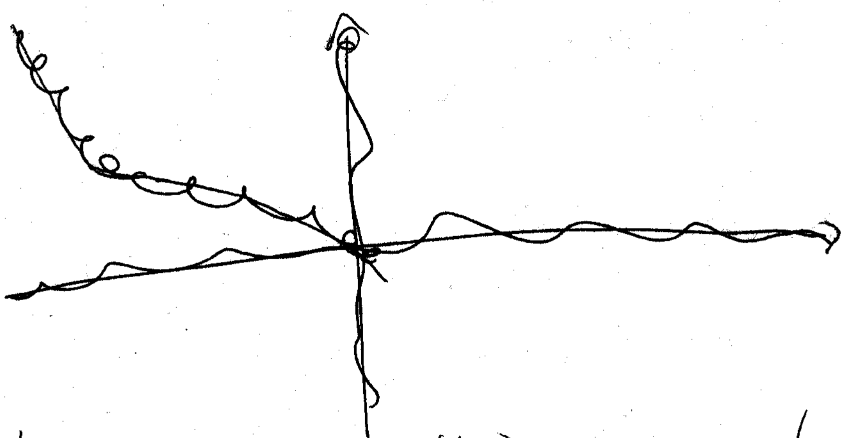


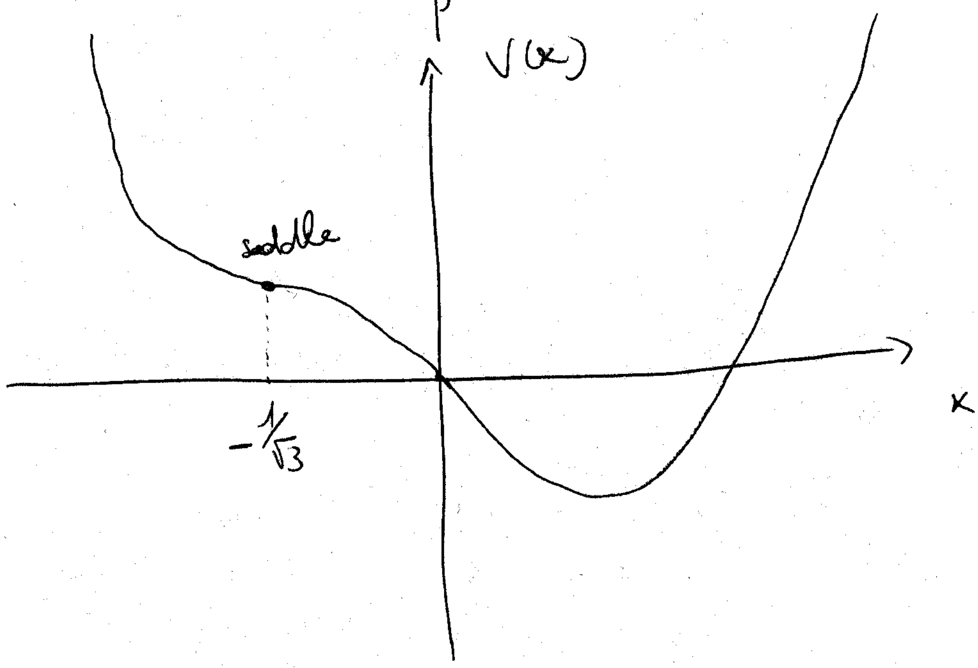
2.7.6

for $r < -\frac{2}{3^{3/2}}$, the fixed point disappears



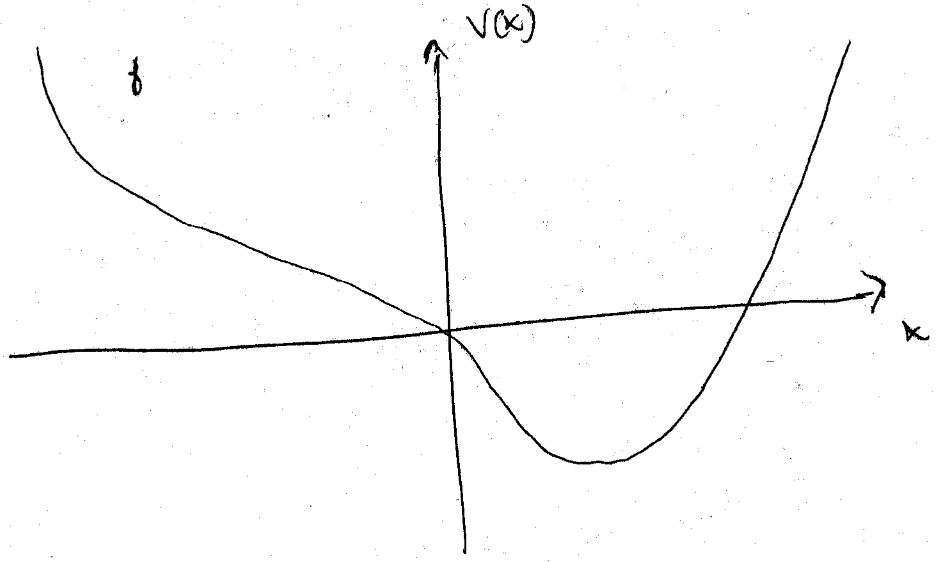
It's possible to do the same reasoning with the negative fixed point, so that if $r = \frac{2}{3^{3/2}}$





2.7.6

and $r > \frac{2}{3^{3/2}}$ yields



To summarize, there are

- 3 fixed points, one stable and two unstable when $|r| < \frac{2}{3^{3/2}}$;
- 1 stable fixed point when $|r| > \frac{2}{3^{3/2}}$;
- 1 stable fixed point and one saddle when $|r| = \frac{2}{3^{3/2}}$.