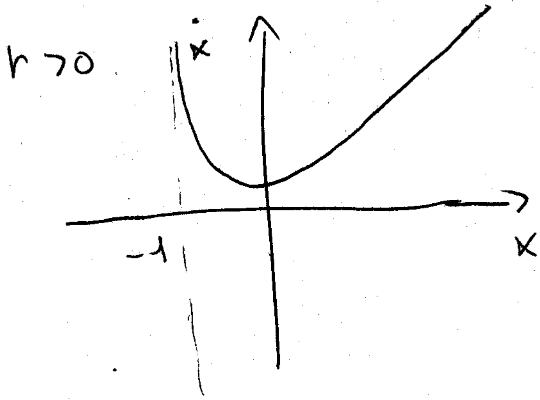
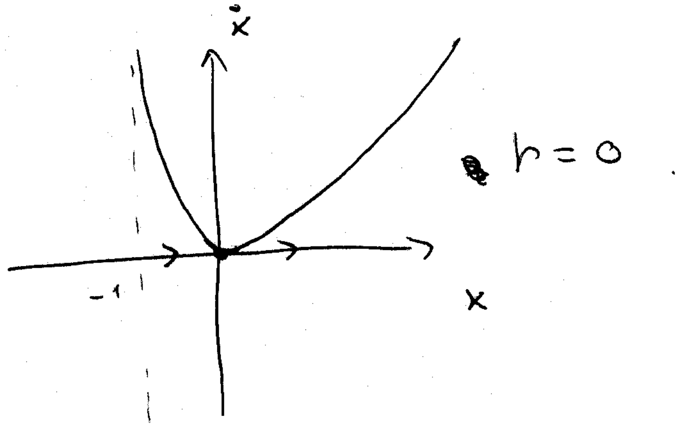
3.1.3

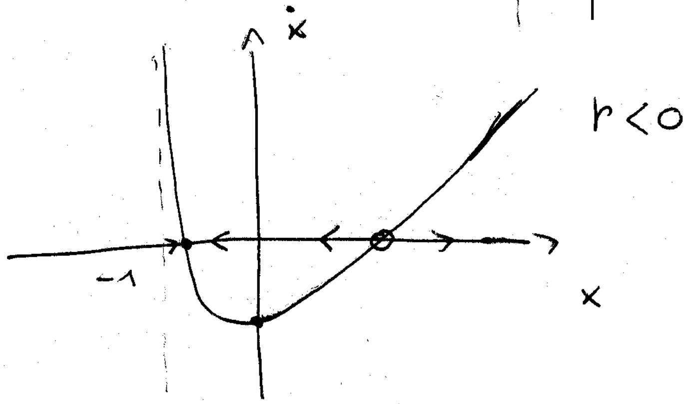
$$\dot{x} = r + x - \ln(1+x) \equiv f(x)$$

$f(x) \to +\infty$ when $x \to +\infty$, $f(x) \to +\infty$ when $x \to -1$

$f'(x) = 1 - \frac{1}{1+x} = 0 \Leftrightarrow x = 0$, so that the vector field looks like this

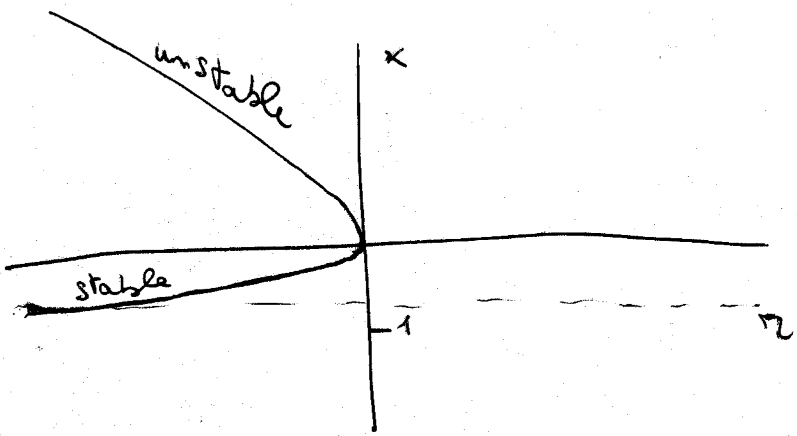






The bifurcation diagram is given by the curve

$$r + x - \ln(1+x) = 0$$



a pair of fixed points (one stable and one unstable) is created as r turns negative, that is we are in the presence of a tangent (or saddle-node) bifurcation.

3.2.6

a) $$x = R\bar{X} - \bar{X}^2 + O(\bar{X}^4) \quad (1)$$

if $$\bar{X} = x + cx^3 + O(x^4) \quad (2)$$

plug (1) into (2) and get

$$\bar{X} = \bar{X} + b\bar{X}^3 + c\left(\bar{X} + b\bar{X}^3\right)^3 + O(\bar{X}^4)$$

$$\Leftrightarrow$$

$$0 = b\bar{X}^3 + c\bar{X}^3 + O(\bar{X}^4)$$

$$\Rightarrow \quad c = -b + O(\bar{X}^4)$$

b) $$\dot{x} = \dot{\bar{X}} + 3b\bar{X}^2\dot{\bar{X}} + O(\bar{X}^4) =$$

$$= R\bar{X} - \bar{X}^2 + a\bar{X}^3 + 3b\bar{X}^2\left(R\bar{X} - \bar{X}^2 + a\bar{X}^3\right) + O(\bar{X}^4)$$

$$= R\bar{X} - \bar{X}^2 + (a + 3bR)\bar{X}^3 + O(\bar{X}^4)$$

$$= R\left(x - bx^3 + O(x^4)\right) - \left(x - bx^3 + O(x^4)\right)^2$$

$$+ (a + 3bR)\left(x - bx^3 + O(x^4)\right)^3 =$$

$$Rx - x^2 + (a + 2bR)x^3 + O(x^4)$$

c) the x^3 term vanishes if $b = -\frac{a}{2R}$

d) $R \neq 0$ is necessary for the transformation to ~~make sense~~ exist if $a \neq 0$.
~~[illegible]~~ If $a = 0$ ~~[illegible]~~ and $R = 0$, then there is no need for it.

3.3.1

a) $\dot{N} = 0 \Rightarrow N = \dfrac{p}{Gn + f}$

and $\dot{n} = \dfrac{pGn}{Gn+f} - kn$

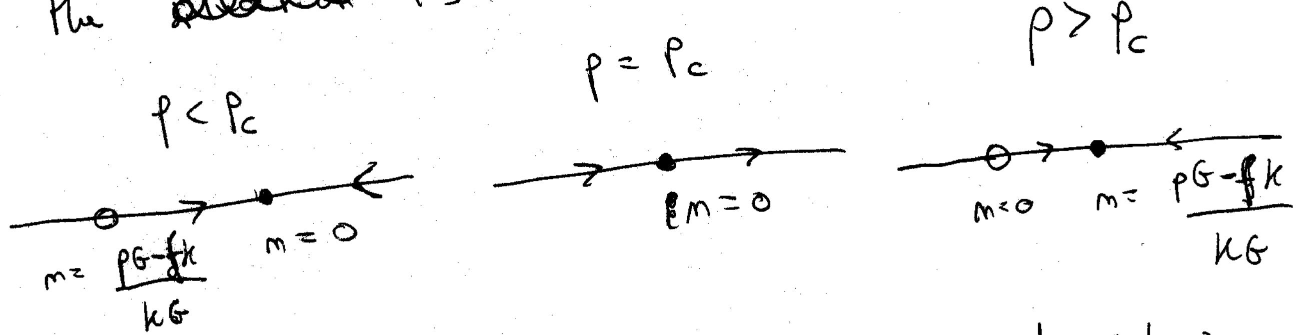
b) $\dot{n} = 0$ iff $n = 0$ or $n = \dfrac{pg - fk}{kG}$

$\left[\dfrac{pGn}{Gn+f} - kn\right]' = \dfrac{fGp}{(Gn+f)^2} - k$, then $n = 0$

the derivative is $\dfrac{Gp}{f} - k > 0$ iff $p > \dfrac{fk}{G} \equiv p_c$

, so that the fixed point $n=0$ is unstable above a certain value of the pump strength

c) the bifurcation is transcritical :



That is our laser turns on when the pump strength is sufficiently high.

d) Rather than a complete analysis, few remarks are made: k is proportional to the decay of n, while f is proportional to the decay of N, hence having $K \ll f$ would help the approximation.
G is proportional to a coupled term (nN), hence it is difficult to compare it to the other parameters.
The pump strength is proportional to the increase in the number of excited atoms, but it's either trivially small (no laser) or above a certain threshold (e.g. $\frac{Kf}{G}$).

HW # 4

3.4.8

$$\dot{x} = rx - \frac{x}{1+x^2} = 0 \quad \text{iff}$$

$$x\left(r + rx^2 - 1\right) = 0 \quad , \text{ that is}$$

$$x = 0 \qquad \text{or} \qquad x^* = \pm\sqrt{\frac{1-r}{r}}$$

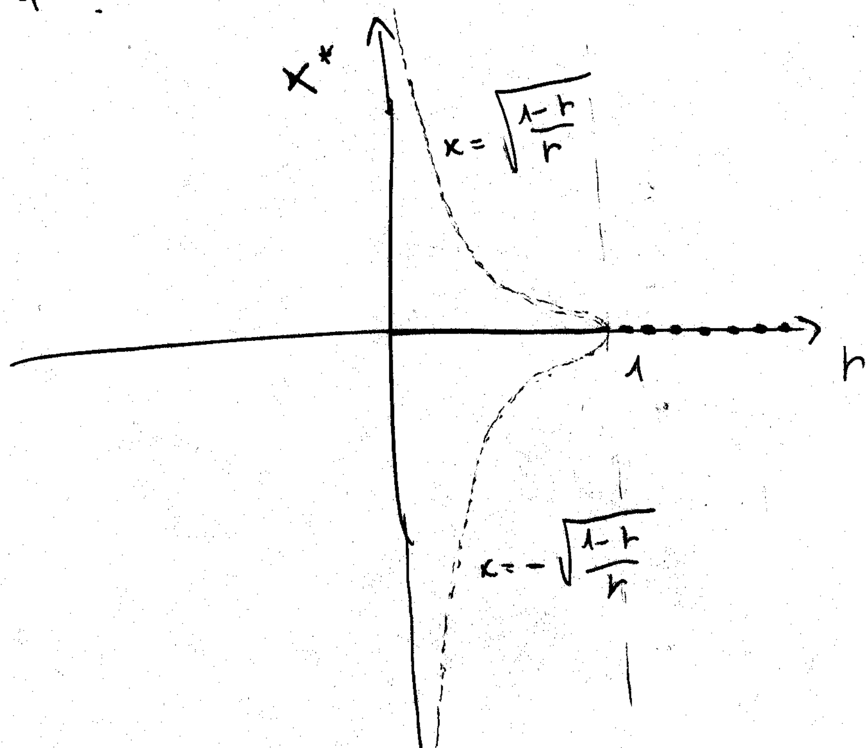
The last two roots exist only if $r < 1$.

$$f'(x) = r - \frac{(1+x^2) - 2x^2}{(1+x^2)^2} \;, \quad f'(x^*) = r - \frac{\frac{2r-1}{r}}{\left(\frac{1}{r}\right)^2} > 0 \text{ iff}$$

$$1 - r > 0 \iff r < 1 .$$

On the other hand, $f'(0) = r - 1 > 0 \iff r > 1$.

Therefore a subcritical pitchfork bifurcation occurs at $r = 1$:



S.7

$$\dot{N} = rN(1 - N/K), \qquad N(0) = N_0$$

a) If N is measured in representatives of a given spicies, then N_0 and hence K are dimensionless. (Alternatively, both could be measured in units of biomass, e.g., kg.) Therefore, $\dot{N}$ has units of $\sec^{-1}$ and so does r.

b) Define $x = N/K$, so $\dot{x} = \frac{\dot{N}}{K} = r\frac{N}{K}\left(1 - \frac{N}{K}\right) = rx(1-x)$

If we define $x_0 = N_0/K$, then $x(0) = N(0)/K = N_0/K = x_0$.

Finally, if we rescale time: $\tau = rt$, then

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{1}{r} r x(1-x) = x(1-x)$$

c) Alternatively, we can choose $u = N/N_0$, so $u(0) = \frac{N(0)}{N_0} = 1$

and $\dot{u} = r\frac{N}{N_0}\left(1 - \frac{N}{N_0}\frac{N_0}{K}\right) = ru\left(1 - \frac{N_0}{K}u\right)$

Defining $\tau = rt$ as before and $k' = \frac{K}{N_0}$, we obtain

$$\frac{du}{d\tau} = \frac{1}{r} ru\left(1 - \frac{1}{k'}u\right) = u(1 - u/k')$$

d) 1^{st} nondimensionalization allows easier interpretation of the results in terms of their dependence on critical conditions, while the second one – dependence on the carrying capacity.