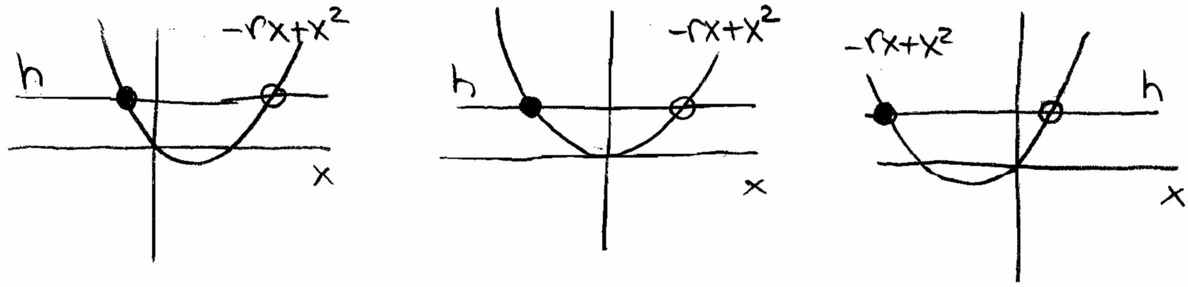
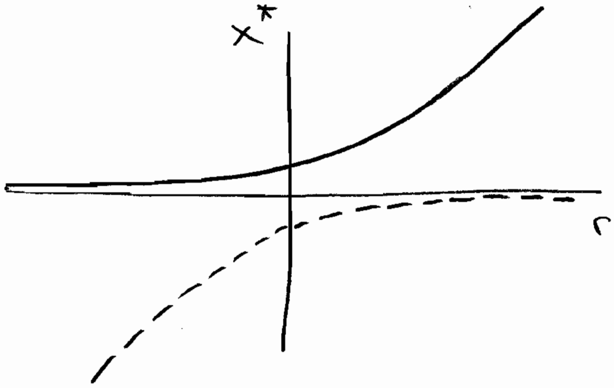
3.6.2.

h > 0:

a)

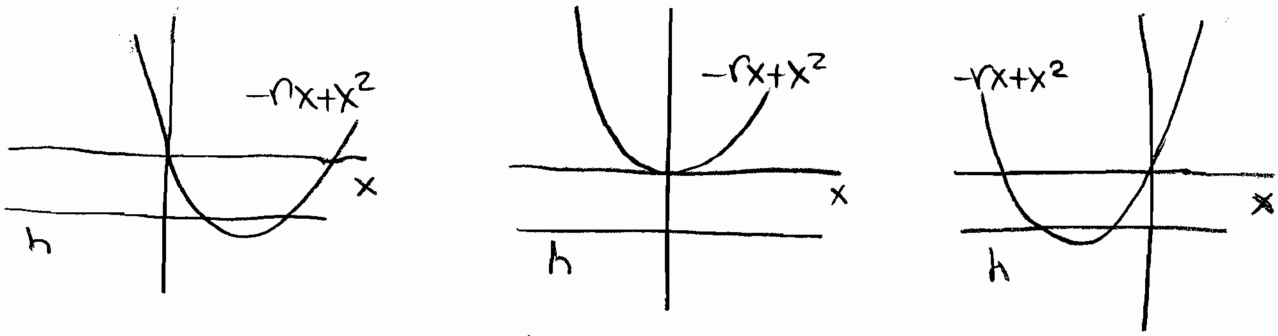


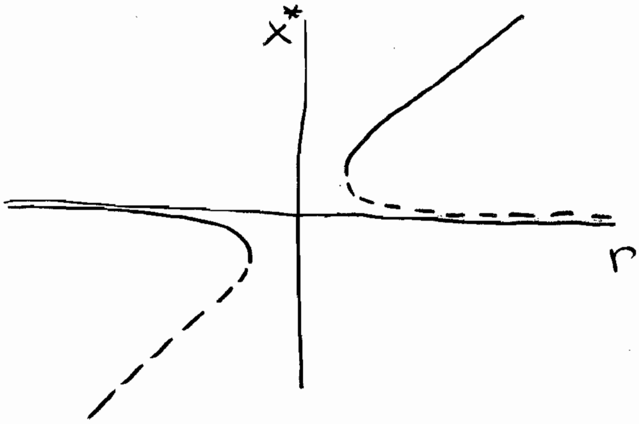
$r<0$ $\quad$ $r=0$ $\quad$ $r>0$



h = 0: Conventional transcritical bifurcation

h < 0:



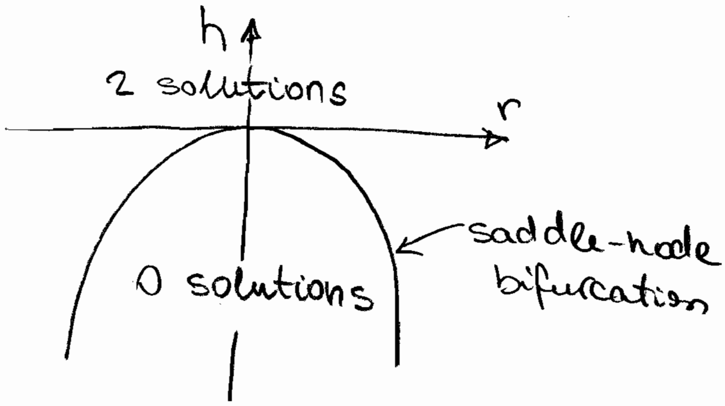


b) Saddle-node bifurcation occurs when

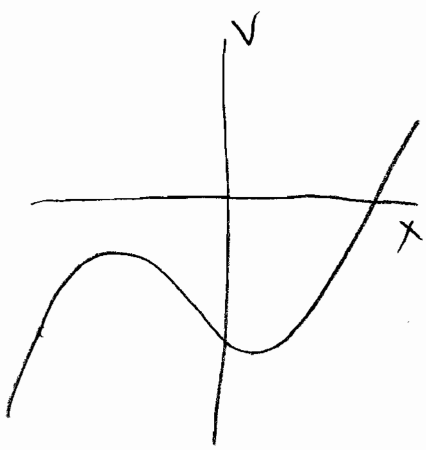
$h + rx - x^2 = 0$ (intersect) and

$\frac{d}{dx}(h + rx - x^2) = 0$ (touch) $\rightarrow r - 2x = 0 \rightarrow x = \frac{r}{2}$

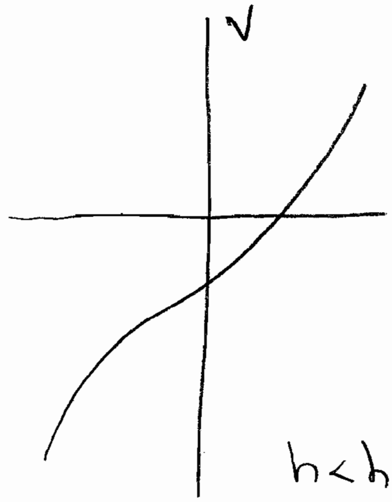
$\Rightarrow h + r\left(\frac{r}{2}\right) - \left(\frac{r}{2}\right)^4 = h + \frac{1}{4}r^2 = 0 \Rightarrow h_c(r) = -\frac{1}{4}r^2$



c) $V = -\int (h + rx - x^2)\,dx = -hx - \frac{1}{2}rx^2 + \frac{1}{3}x^3$ (+ const)



$h > h_c(r)$



$h < h_c(r)$

3.7.6

a) $\dot{x} + \dot{y} + \dot{z} = -kxy + kyx + ly - ly = 0$

$\Rightarrow x + y + z = C \equiv N$

b) $\dot{x} = -kxy = -kx\frac{\dot{z}}{l} \Rightarrow \frac{\dot{x}}{x} = -\frac{k}{l}\dot{z}$ integrate in dx and dz

and get $x(t) = x_0 e^{-k/l\, z(t)}$

c) $\dot{z} = ly = l\left[N - x - z\right] = l[N - x_0 e^{-k/l\, z} - z$

d) $\dot{z} = l\left[N - z - x_0 e^{-k/l\, z}\right]$

$\Rightarrow \frac{1}{l x_0}\dot{z} = \frac{N}{x_0} - \frac{z}{x_0} - e^{-k/l\, z}$

call $u = \frac{k}{l} z$ and get

$\frac{1}{k x_0}\dot{u} = \frac{N}{x_0} - \frac{l}{k x_0} u - e^{-u}$

Now let $\tau = x_0 k t$

$\Rightarrow \frac{du}{d\tau} = \underset{= a}{\frac{N}{x_0}} - \underset{= b}{\frac{l}{k x_0}} u - e^{-u}$

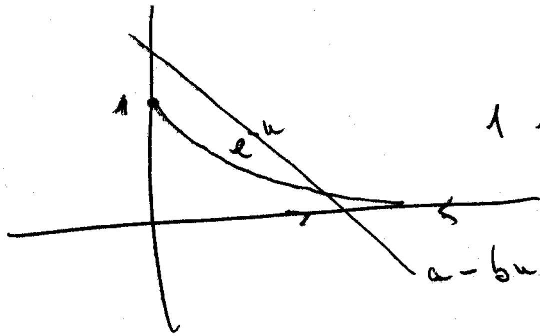
e) $a = \frac{N}{x_0}$ is 1 if everybody is healthy at time $t=0$ $(N = x_0)$

or else it's ~~$\frac{x+y+z}{x_0}$~~ $\frac{x+y+z}{x_0} > 1$

$b = \frac{l}{h x_0}$ everything positive, hence $b > 0$

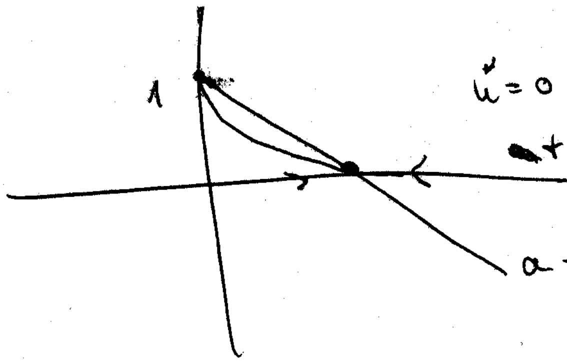
f) the condition for fixed points is $a - bu - e^{-u} = 0$

i)



1 fixed point, stable

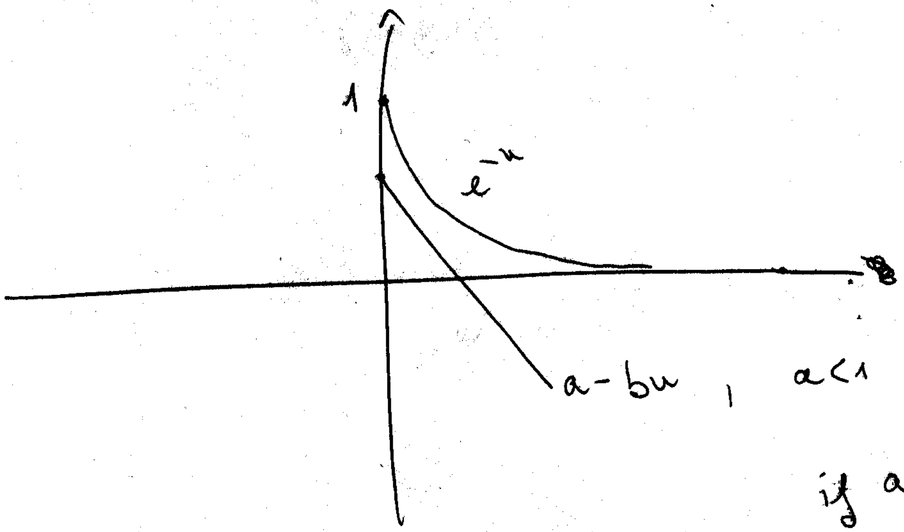
ii)



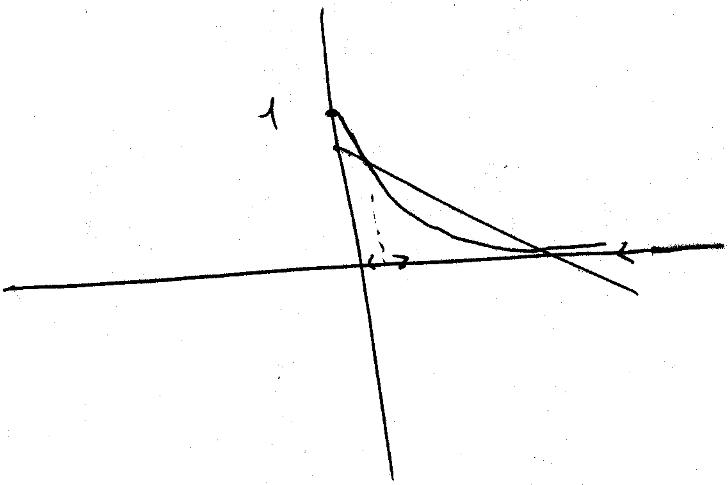
$u^* = 0$, unstable

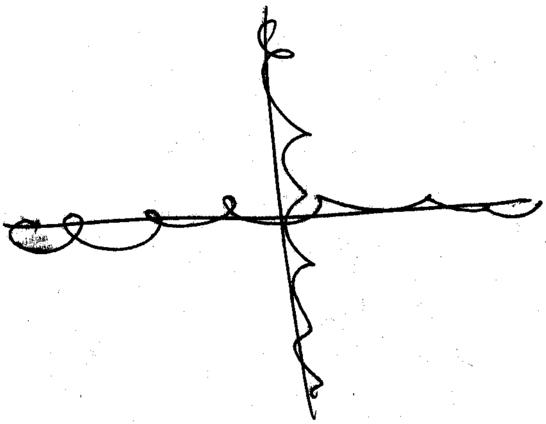
+ another stable fixed point

iii)



if $a<1$, there can be either no fixed points or one stable and one unstable fixed points depending on b





g) $\frac{d}{dt} z = \frac{1}{kx_0} \frac{d}{dt} u = l y$

So that all three functions have the same extrema at the same time.

h) ~~...~~ $\frac{1}{kx_0}\frac{d}{dt}u = a - bu - e^{-u}$

Now the maximum for that function can be found by differentiating $a - bu - e^{-u}$: $f'(u) = 0$

$\Leftrightarrow -b + e^{-u} = 0 \Leftrightarrow u = -\ln b$

u makes sense only if positive, that is that maximum only exists if $\ln b < 0 \Leftrightarrow b < 1$,

i) recall that $z(0) = 0$ hence $u(0) = 0$, so that if $b > 1$ $-b + e^{-u} < 0$ at all times, that is $\frac{du}{dt}$ is decreasing from $u(0) = 0$.

j) $b = \frac{l}{kx_0}$, ~~...~~ It's > 1 if the death rate is greater than the infection rate times the initial number of healthy people, that is it takes a ~~high lifeless~~ relatively high infection rate for the epidemic to occur.

k) the main problem w/ AIDS is that infection does not mean illness, so an additional term in the equation would be needed to describe the rate at which infected people fall sick. Also death rate should be time-dependent, due to the progress of medicine

4.1.8

a) $V(\theta) = -\sin\theta + C_1$

now $V(\theta + 2k\pi) = -\sin(\theta + 2k\pi) + C_1 = -\sin\theta + C_1$

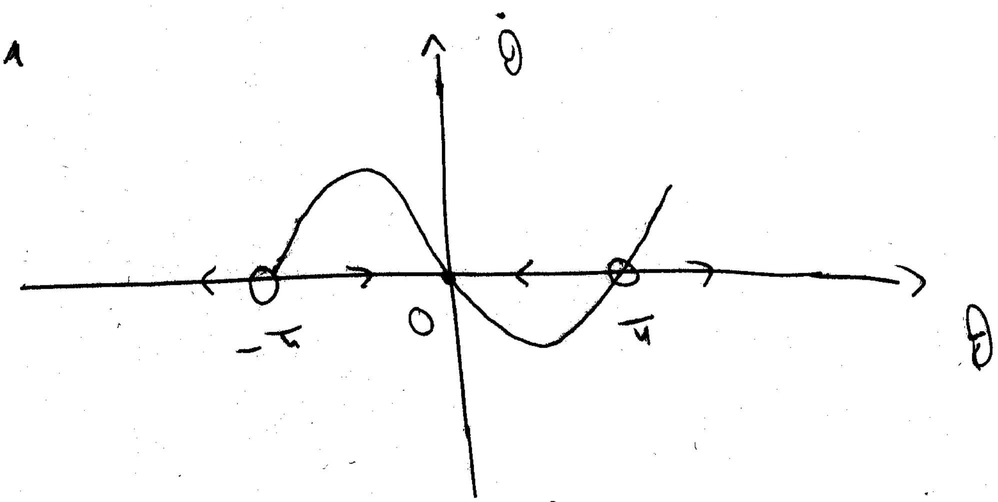
then it's single-valued

b) $V(\theta) = \theta + C_1$, but $V(\theta + 2k\pi) \neq V(\theta)$

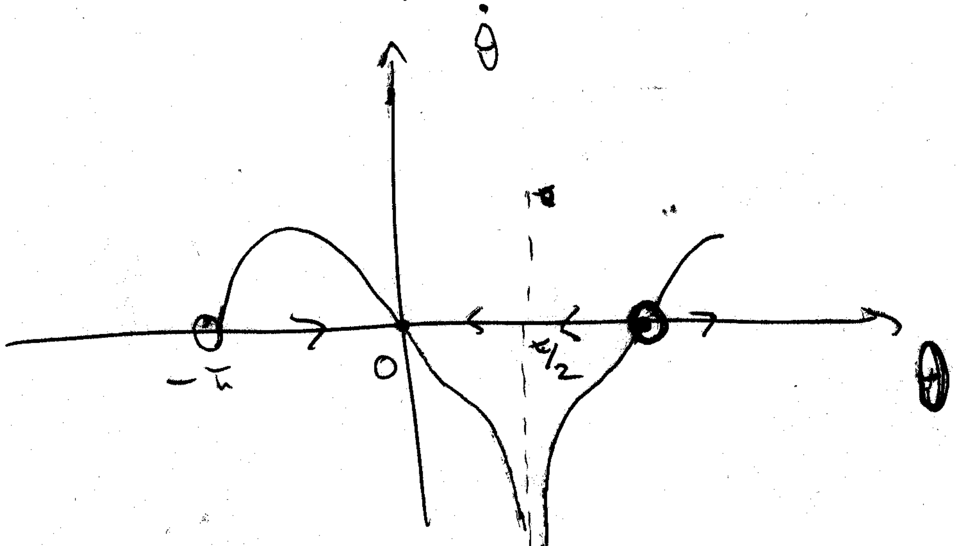
c) the general rule is $V(\theta + 2k\pi) = V(\theta)$.

4.3.7 $\dot{\theta} = \dfrac{\sin\theta}{\mu + \sin\theta}$

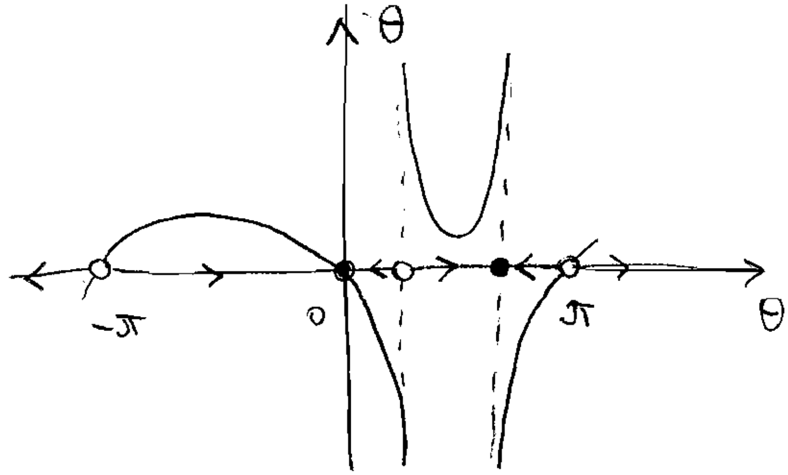
i) $\mu \leq -1$



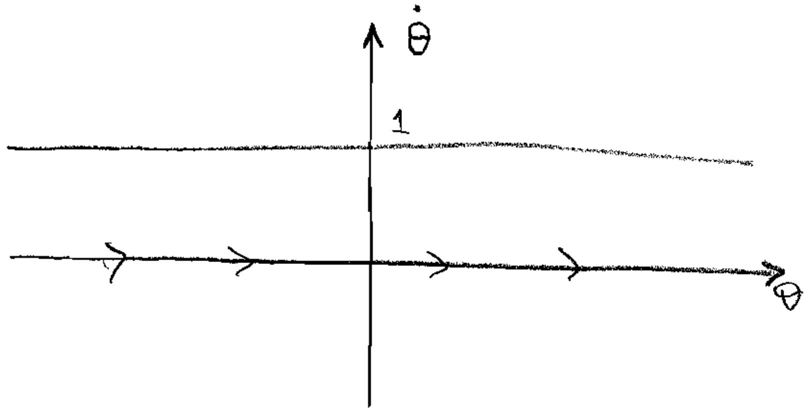
ii) $\mu = -1$



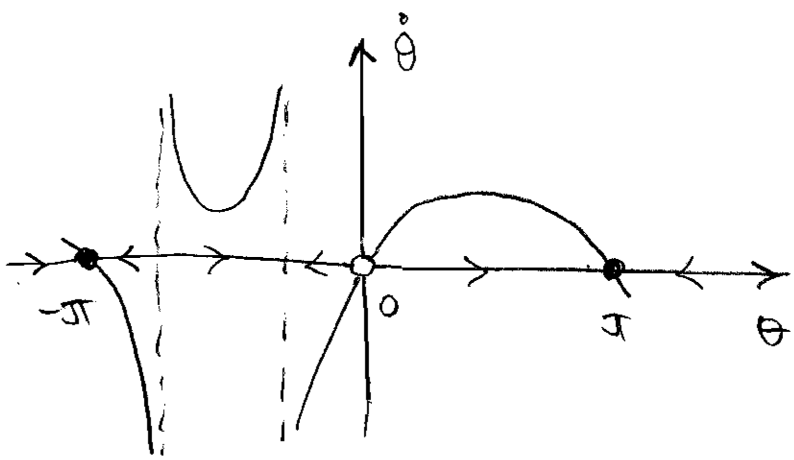
iii) $-1<\mu<0$



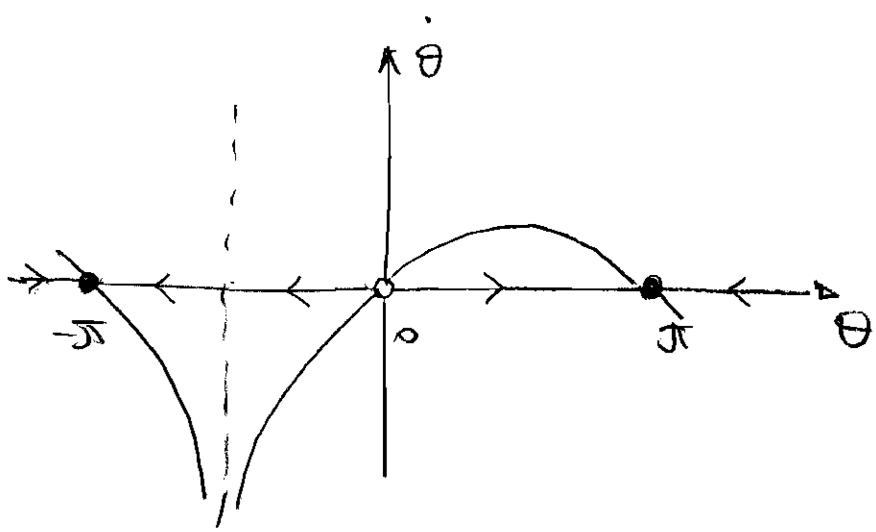
iv) $\mu=0$



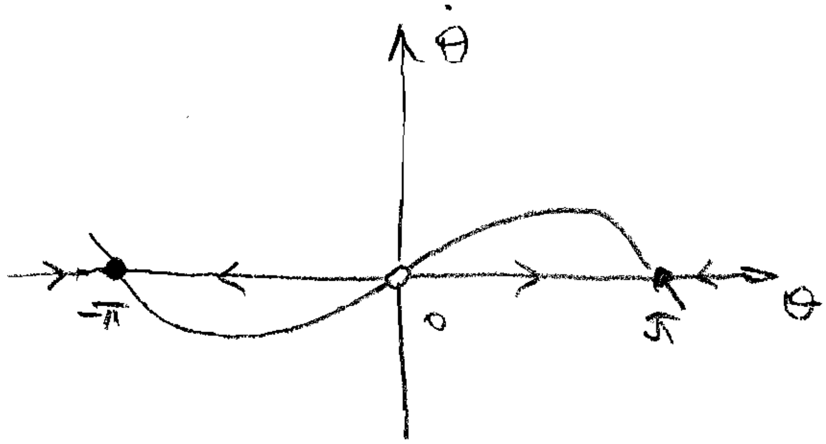
v) $0<\mu<1$



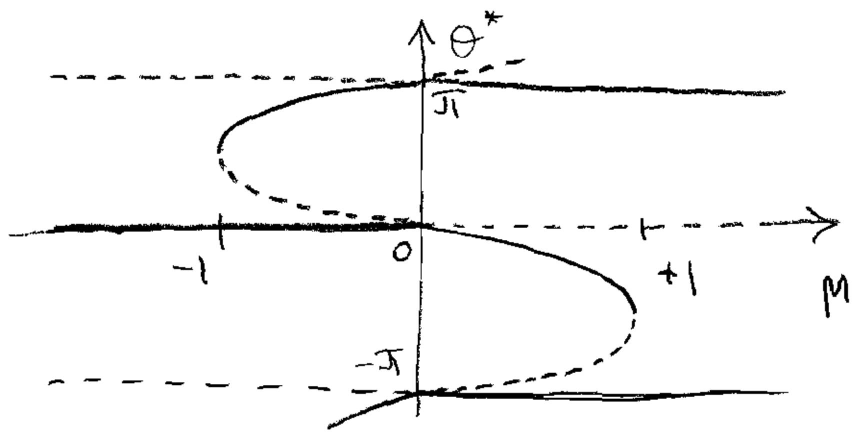
vi) $\mu=1$



vii)



The bifurcation diagram looks like this:



From the bifurcation diagram it looks like the bifurcations at $\mu = \pm 1$ are of saddle-node type and the bifurcation at $\mu = 0$ is of transcritical type. However, neither conclusion is correct, as can be seen by noting that $f(\theta,\mu) = \dfrac{\sin\theta}{\mu + \sin\theta}$ does not have a Taylor expansion at either $\mu = 0$ or $\mu = \pm 1$!

The "fixed points" at $\theta = -\arcsin\mu$ are not true fixed points since $\dot\theta \neq 0$ there. However, the behavior of solutions in their vicinity is just like that near the true fixed points.

4.6.3

$$\dot{\phi} = I/I_c - \sin\phi$$

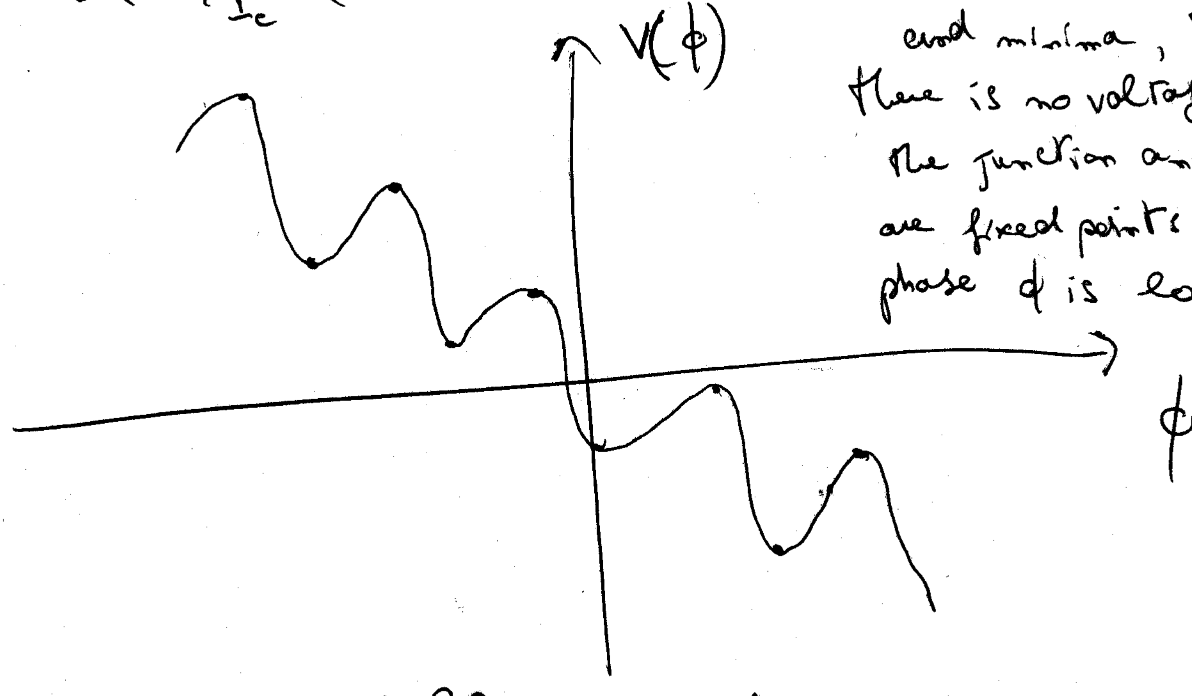
a)

$$\Rightarrow \quad V(\phi) = -I/I_c\,\phi - \cos\phi + C'$$

which contains the function $I/I_c\,\phi$, that is not single-valued (see problem 4.1.8), so that the potential itself is not single-valued.

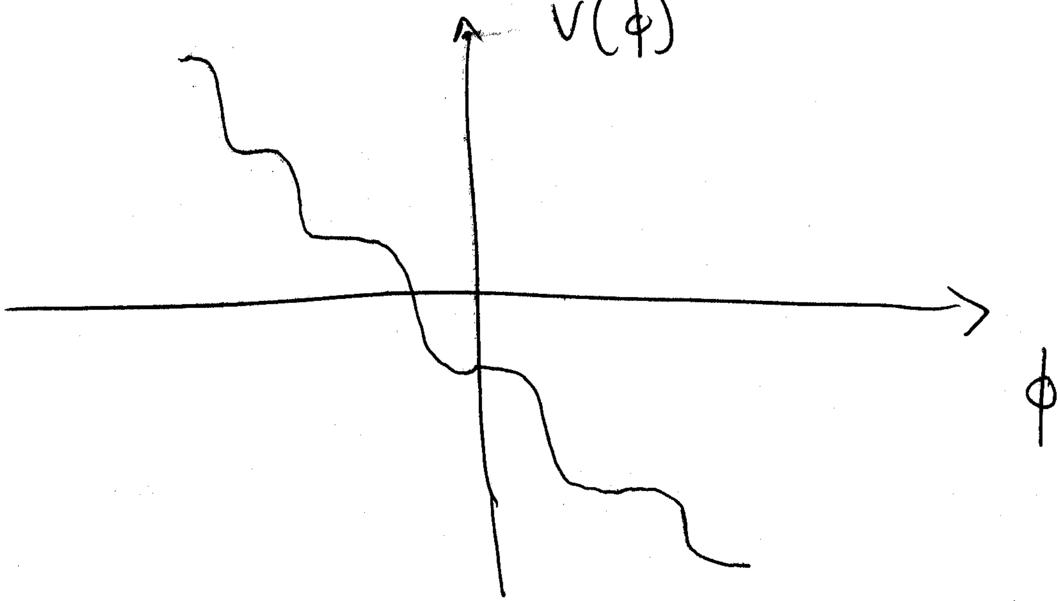
b) I is the ~~absolute~~ magnitude of the current, hence it's always positive. There are three main pictures:

i) $0 < I/I_c < 1$



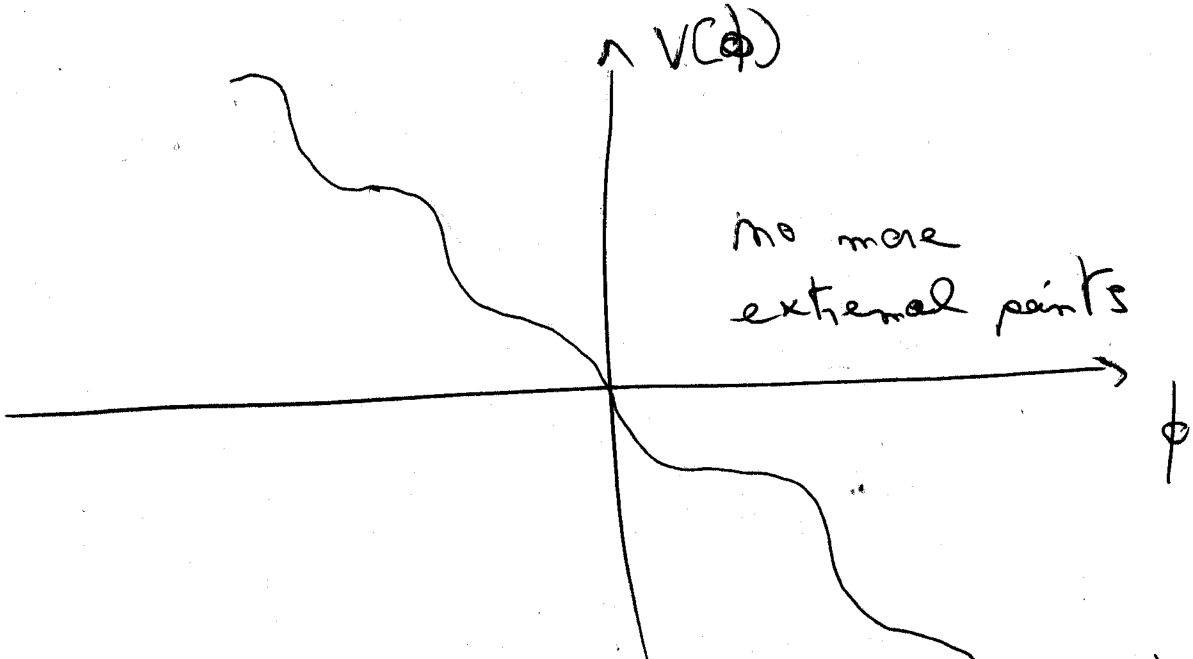
There are maxima and minima, that is there is no voltage across the junction and there are fixed points where the phase ϕ is locked.

ii) $I = I_c$



the previous fixed points become saddles.

iii) $I/I_c > 1$



c) As I becomes larger than I_c, the junction develops its own voltage, so that it's impossible to lock the phase ϕ.