5.1.10

HW #6

a) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ eigenvalues are $\lambda = \pm 2i$

meaning the origin is a center, hence it's Liapunov stable.

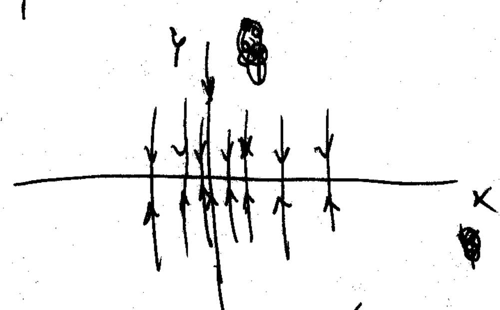
b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ eigenvalues are $\lambda = \pm \sqrt{2}$

Therefore the origin is a saddle, unstable in general.

c) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ solution is
$x = C_1'$
$y = C_1' t + C_1$

starting any close to the origin, $|y|$ increases as time goes by, therefore the point is unstable.

d) solution is $x = C_1'$
$y = C_1 e^{-t}$



meaning starting close to the origin at (ϵ, ϵ) leads to $(\epsilon, 0)$, which is closer, that is the origin is Liapunov stable.

e) solution is $x = C_1 e^{-t}$
$y = C_2 e^{-5t}$, that is the origin is asymptotically stable.

f) solution is $\begin{aligned} x &= c_1' e^{t} \\ y &= c_2' e^{t} \end{aligned}$, that is the origin is unstable.

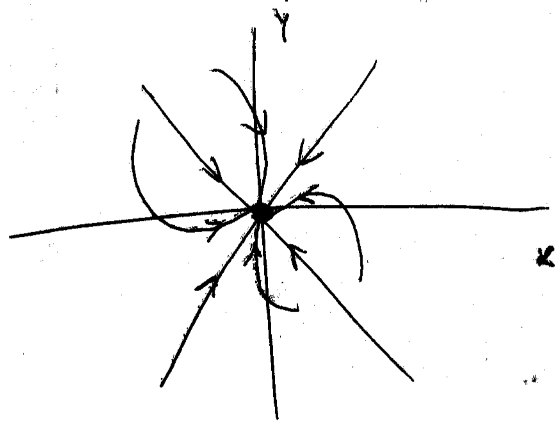
5.2-13

a) $\begin{cases} \dot{x} = y \\ m\dot{y} + by + kx = 0 \end{cases}$

b) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues are $\lambda = \dfrac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}}{2}$

Square root is real iff $b^2 > 4km$
in which case both the eigenvalues are real and negative,
that is the equilibrium point is a stable node.



fast and slow ~~speed~~ directions depend on the parameters b, k, m.

if $b^2 = 4km$ there are two equal eigenvalues,

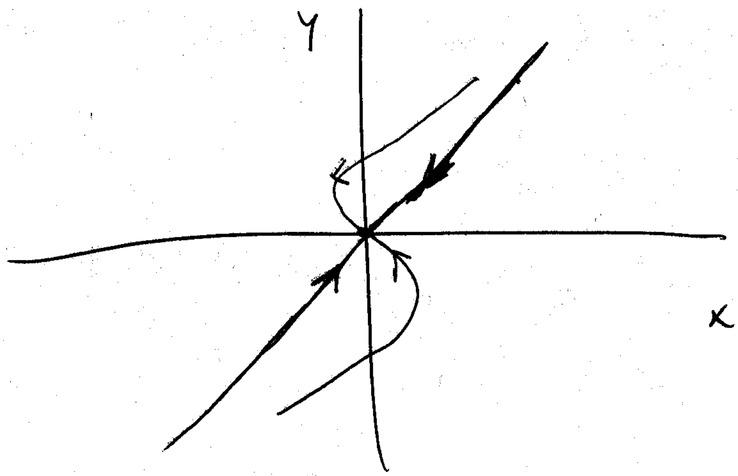
$\lambda = -\dfrac{b}{2m}$, on the other hand the matrix

$$\begin{pmatrix} 0-\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m}-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} y = -\dfrac{b}{2m} x \\ -\dfrac{k}{m} x - \dfrac{b}{2m} y = 0 \quad (b^2 = 4km) \Rightarrow y = -\dfrac{b}{2m} x \end{cases}$$

which defines a one-dimensional subspace,

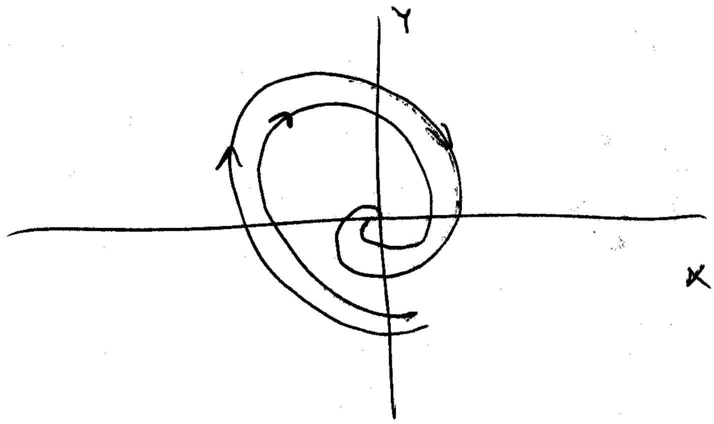
that is we are in the presence of a degenerate node:



if $\frac{b^2}{m} < 4km$ there are two complex eigenvalues

both with negative real part : $\lambda = \dfrac{-\frac{b}{m} \pm i\sqrt{\left|\frac{b^2}{m^2} - \frac{4k}{m}\right|}}{2}$

therefore the fixed point at the origin is a stable spiral :



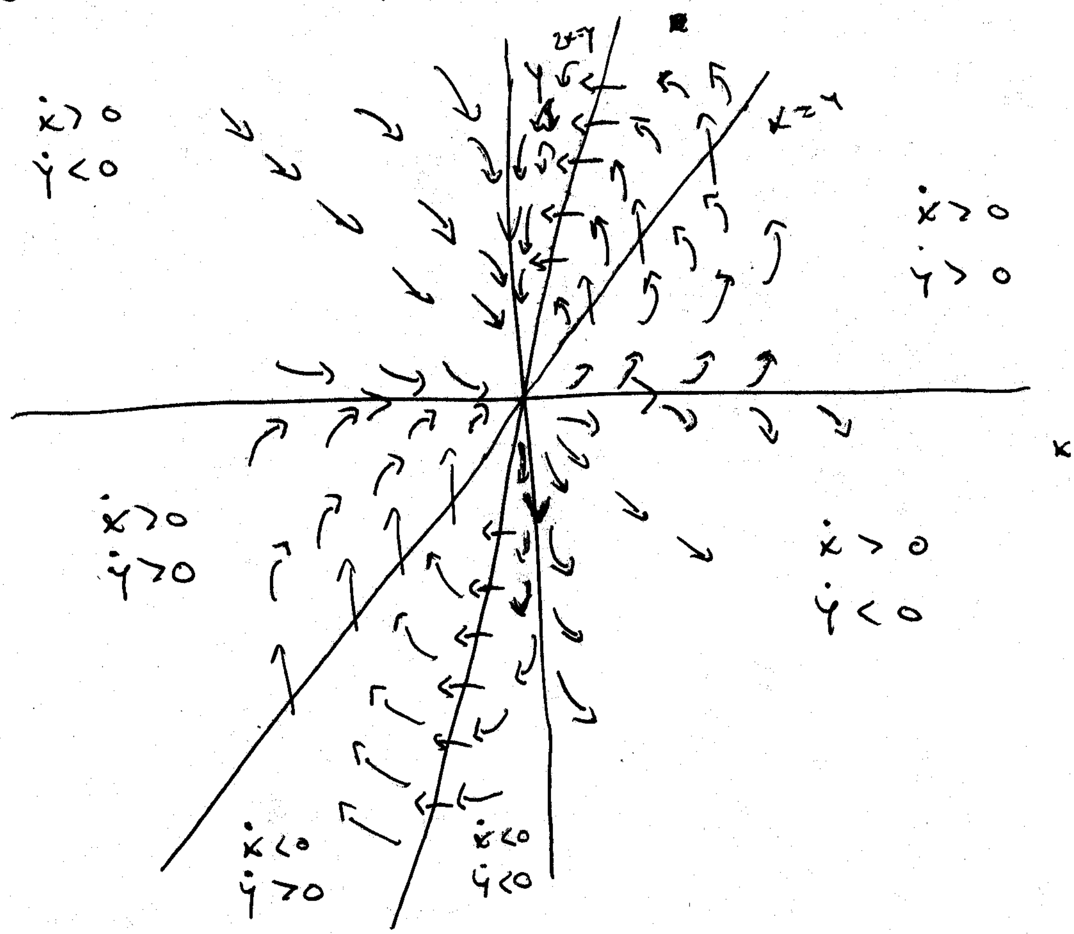
c) The spiral corresponds to underdamped vibrations of the spring, when the friction is relatively weak and the spring can still oscillate. The critical damping takes place when $b^2 = 4km$, ~~[illegible]~~ and it represents the transition to the overdamping ($b^2 > 4km$), when the spring doesn't ~~oscillate~~ vibrate and is immediately stopped by the friction.

6.1.3.

$$\begin{cases} \dot{x} = x(x-y) \\ \dot{y} = y(2x-y) \end{cases}$$

fixed point at the origin $x^* = (0,0)$,

nullclines $x = y \Rightarrow \dot{x} = 0$ and $2x = y \Rightarrow \dot{y} = 0$



6.2.1

A fixed point is itself an independent trajectory. The trajectories that appear to intersect it, only approach it in an infinite time, therefore there is no contradiction since there is no real intersection.

6.3.1. $\begin{cases} \dot{x} = x - y = 0 \\ \dot{y} = x^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 2 \\ y = x = \pm 2 \end{cases}, \quad J = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$

$(x^*, y^*) = (-2,-2)$: $\quad J = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}, \quad \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{17})$

$\vec{e}_\pm = \left(\left(\frac{1}{2}(1 \mp \sqrt{17}) \right)^{-1}, 1 \right)$

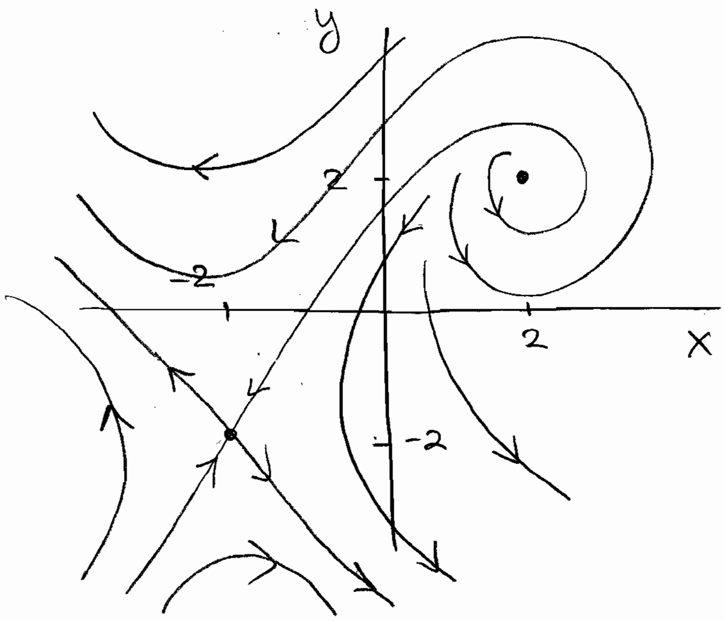
saddle

$\lambda_+ > 0, \quad \vec{e}_+ \simeq (-0.64, 1)$ – unstable

$\lambda_- < 0, \quad \vec{e}_- \simeq (0.39, 1)$ – stable

$(x^*, y^*) = (2,2)$: $\quad J = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}, \quad \lambda_\pm = \frac{1}{2}(1 \pm i\sqrt{15})$

unstable spiral, $Re(\lambda_\pm) > 0$



6.4.7

$$\dot{n}_1 = G_1 N n_1 - k_1 n_1 = G_1(N_0 - \alpha_1 n_1 - \alpha_2 n_2) n_1 - k_1 n_1$$

$$\dot{n}_2 = G_2 N n_2 - k_2 n_2 = G_2(N_0 - \alpha_1 n_1 - \alpha_2 n_2) n_2 - k_2 n_2$$

the Jacobian matrix is

$$J = \begin{pmatrix} G_1 N_0 - 2G_1 n_1 - G_1 \alpha_2 n_2 - k_1 & -G_1 \alpha_2 n_1 \\ -G_2 \alpha_1 n_2 & G_2 N_0 - \alpha_1 G_2 n_1 - 2G_2 \alpha_2 n_2 - k_2 \end{pmatrix}$$

at the fixed point $\vec{n} = (0,0)$ the Jacobian reads

$$J(0,0) = \begin{pmatrix} G_1 N_0 - k_1 & 0 \\ 0 & G_2 N_0 - k_2 \end{pmatrix}$$

so that

i) if $\begin{cases} G_1 N_0 > k_1 \\ G_2 N_0 > k_2 \end{cases}$ the origin is an unstable node (star node if $k_1 = k_2$ and $G_1 = G_2$)

ii) if $\begin{cases} G_1 N_0 > k_1 \\ G_2 N_0 < k_2 \end{cases}$ or vice versa, it's a saddle.

iii) if $\begin{cases} G_1 N_0 < k_1 \\ G_2 N_0 < k_2 \end{cases}$ it's a stable node (again, star node if $k_1 = k_2$ and $G_1 = G_2$)

Physical meaning: the laser is turned off.

iv) if $\begin{cases} G_1 N_0 = k_1 \\ G_2 N_0 \gtrless k_2 \end{cases}$ or viceversa, we have marginal stability at the origin.

v) if $\begin{cases} G_1 N_0 = k_1 \\ G_2 N_0 = k_2 \end{cases}$ the origin is a center.

b) consider the fixed point

$$n_1 = 0 \quad , \quad n_2 = \frac{G_2 N_0 - k_2}{G_2 \alpha_2}$$

that comes from setting $n_1 = 0$ and solving $\dot{n}_2 = 0$.

At this point the Jacobian reads

$$J = \begin{pmatrix} \frac{G_1 k_2}{G_2} & 0 \\ -\frac{\alpha_1 (G_2 N_0 - k_2)}{\alpha_2} & k_2 - G_2 N_0 \end{pmatrix}$$

the eigenvalues are

$$\lambda_1 = \frac{G_1}{G_2} k_2 \quad \text{and} \quad \lambda_2 = G_2 N_0 - k_2$$

Now, $\lambda_1 > 0$ always, therefore this point is never stable. Actually $G_2 N_0$ must be $> k_2$ in order for n_2 to be positive, therefore it's an unstable node.

the same reasoning applies to ~~the~~ the fixed point

$$n_1 = \frac{G_1 N_0 - K_1}{G_1 \alpha_1}, \quad n_2 = 0$$

that is easily calculated by setting $n_2 = 0$ and solving $\dot{n}_1 = 0$. In this case we obtain two always positive eigenvalues.

Rather than searching for all the fixed points of the system, let's focus on its ~~f~~ physical meaning: it's a laser, and in order for it to be on, we need to find at least one fixed point (not the origin), that ~~is~~ be stable, so that any initial condition ~~other~~ ~~except for the other~~ in the neighborhood will ~~con~~ converge to a positive number of photons, n_1^* and n_2^*.

Let's look for a point such that $n_1 = n_2$, in which case $\dot{n}_1 = 0 \iff n_1 = 0$ or $n_1 = \dfrac{G_1 N_0 - K_1}{G_1(\alpha_1 + \alpha_2)}$

that also verifies $\dot{n}_2 = \dot{n}_1 = 0$ only if

$$K_1 = K_2 \quad \text{and} \quad G_1 = G_2 .$$

the Jacobian matrix reads

$$J = \begin{pmatrix} k_1 - \alpha_1\left[\dfrac{G_1N_0 - k_1}{\alpha_1+\alpha_2}\right] & -\alpha_2\left[\dfrac{G_1N_0 - k_1}{\alpha_1+\alpha_2}\right] \\ -\alpha_1\left[\dfrac{G_1N_0 - k_1}{\alpha_1+\alpha_2}\right] & k_1 - \alpha_2\left[\dfrac{G_1N_0 - k_1}{\alpha_1+\alpha_2}\right] \end{pmatrix}$$

the eigenvalues of this matrix are

$$\lambda_{1,2} = \frac{K_1 - G_1N_0 \pm \sqrt{G_1^2N_0^2 + 2G_1N_0k_1 - 7K_1^2}}{2}$$

let's consider the case $(G_1N_0 > k_1)$ in which the origin is unstable : one of the two eigenvalues above is certainly negative, in order for the other one to be negative we need

$$k_1 - G_1N_0 < -\sqrt{G_1^2N_0^2 + 2G_1N_0k_1 - 7k_1^2}$$

$$\Leftrightarrow \quad k_1 < \frac{G_1N_0}{2}$$, so that for that condition the point $\vec{n} = \left(\dfrac{G_1N_0 - k_1}{G_1(\alpha_1+\alpha_2)}, \dfrac{G_1N_0 - k_1}{G_1(\alpha_1+\alpha_2)}\right)$ is a stable node, meaning the number of photons in the laser in the stationary regime.