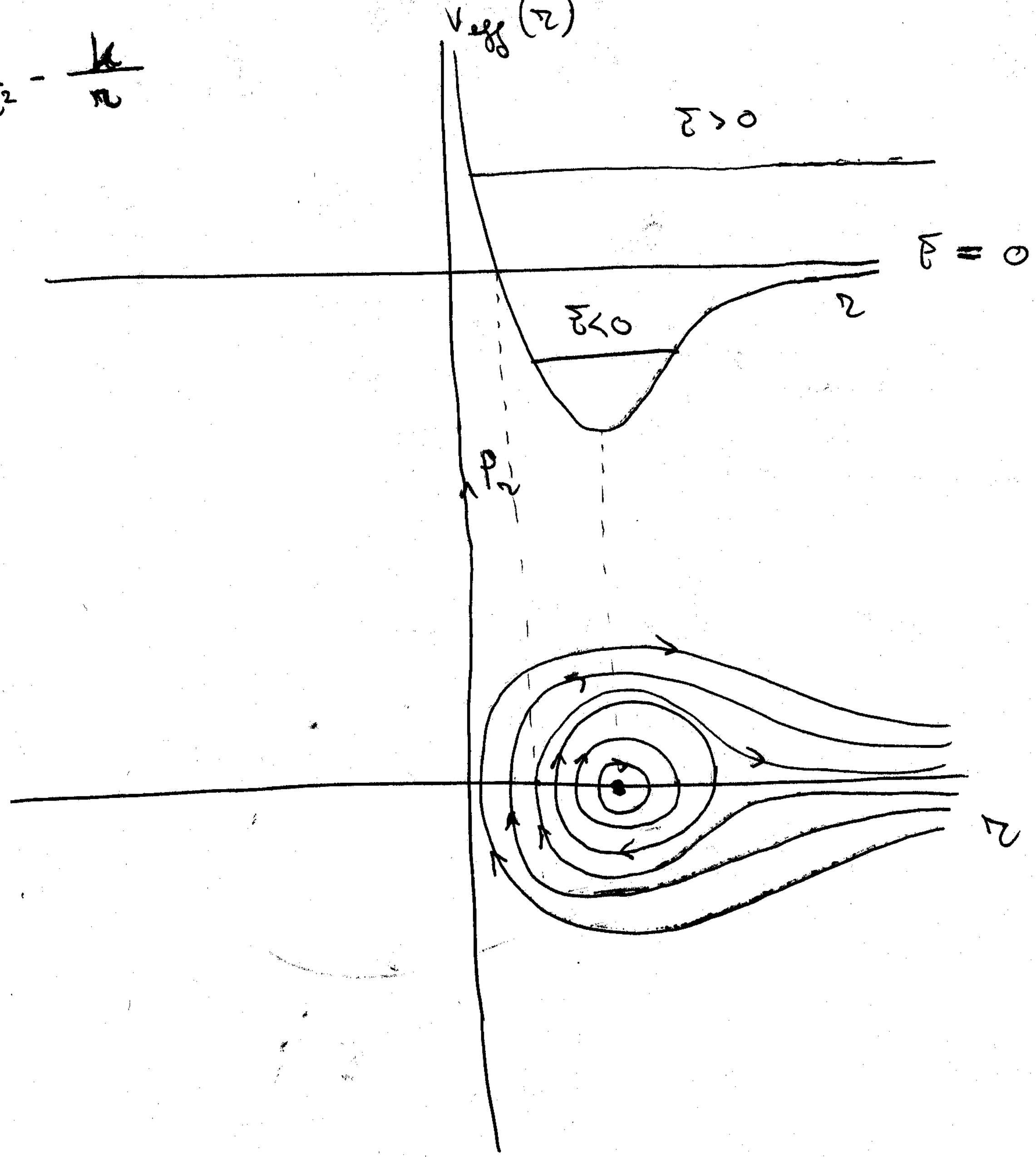
6.5.10

HW # 7

$$V_{eff} = \frac{h^2}{2r^2} - \frac{k}{r}$$

a)



b) From the previous picture we can see that the trajectories are not closed if $E \geq 0$. Let's look for the minimum of the effective potential:

$$V'_{eff}(r) = -\frac{h^2}{r^3} + \frac{k}{r^2} = 0 \iff r_c = \frac{h^2}{k}$$

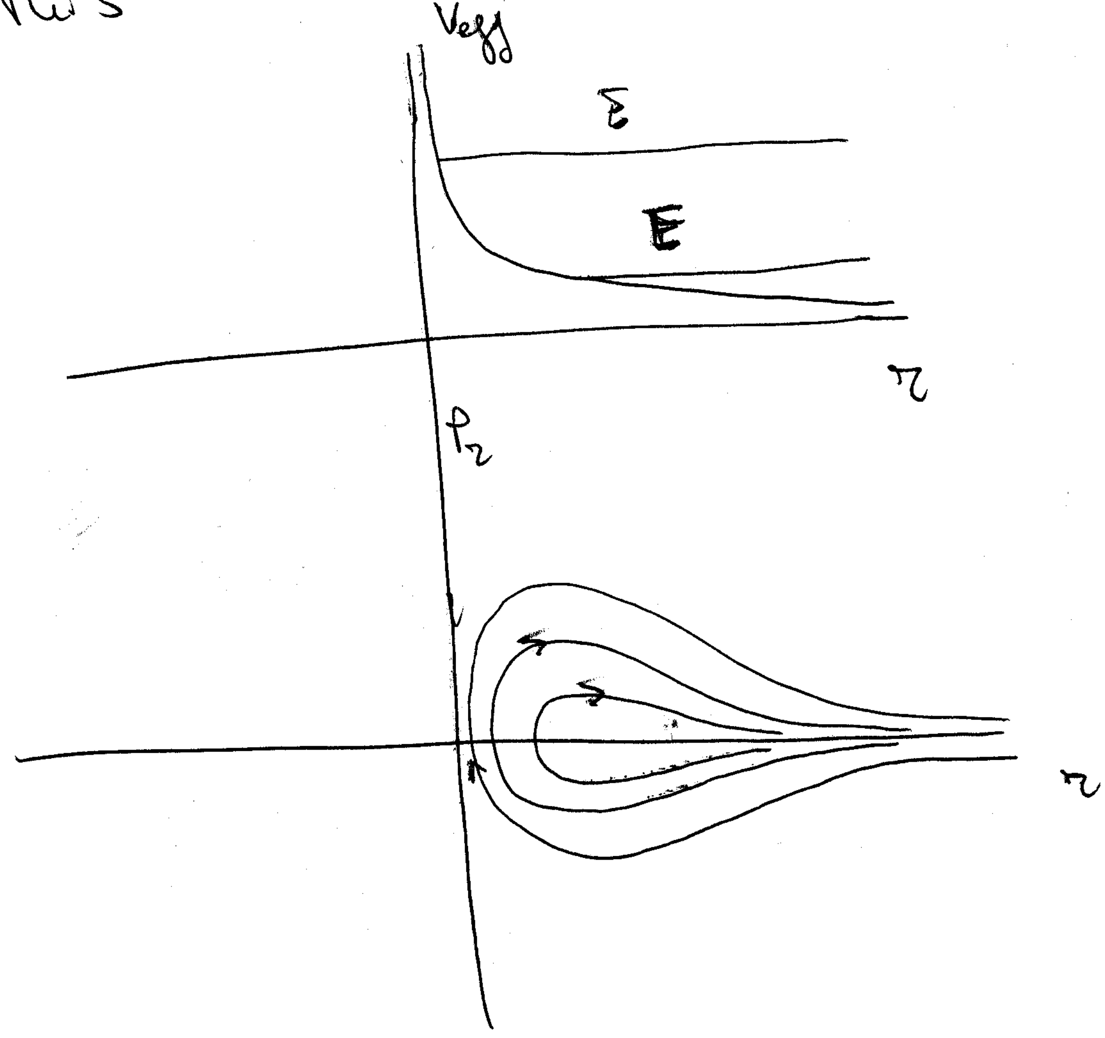
That represents a circular orbit, that is $\dot{r} = 0$ and therefore $p_{r_c} = 0$, so that the $H(p, r)$ reads

$$H(p, r_c) = \frac{\cancel{h^2} h^2}{2 h^{\cancel{4} 2}} - \frac{K^2}{h^2} = E \Rightarrow E = -\frac{K^2}{2h^2}$$

minimum value of the energy for which we have closed ~~orbits~~ trajectories

c) If $K < 0 \Rightarrow V_{eff}(r) = \frac{h^2}{2r^2} + \frac{|K|}{r}$

which has no extremum points, so that it looks like this



6.6.3.

a) $t \to -t \Rightarrow y \to -y$

$\frac{dx}{d(-t)} = -\dot{x} = -\sin y = \sin(-y)$ ✓ $\quad \frac{d(-y)}{d(-t)} = \dot{y} = \sin x$ ✓

b) $\begin{cases} \dot{x} = \sin y = 0 \\ \dot{y} = \sin x = 0 \end{cases} \Rightarrow \begin{cases} x^* = \pi n \\ y^* = \pi m \end{cases}, \quad J = \begin{pmatrix} 0 & \cos y \\ \cos x & 0 \end{pmatrix}$

c) $\begin{cases} u = x+y \\ v = x-y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$

$\dot{u} = \dot{x} + \dot{y} = \sin x + \sin y = \sin\frac{u+v}{2} + \sin\frac{u-v}{2} = 2\sin\frac{u}{2}\cos\frac{v}{2}$

$\dot{u} = 0$ for $u = 0 = x+y$ (or $y = -x$)

$\dot{v} = \dot{x} - \dot{y} = \sin y - \sin x = \sin\frac{u-v}{2} - \sin\frac{u+v}{2} = -2\sin\frac{v}{2}\cos\frac{u}{2}$

$\dot{v} = 0$ for $v = 0 = x-y$ (or $y = x$)

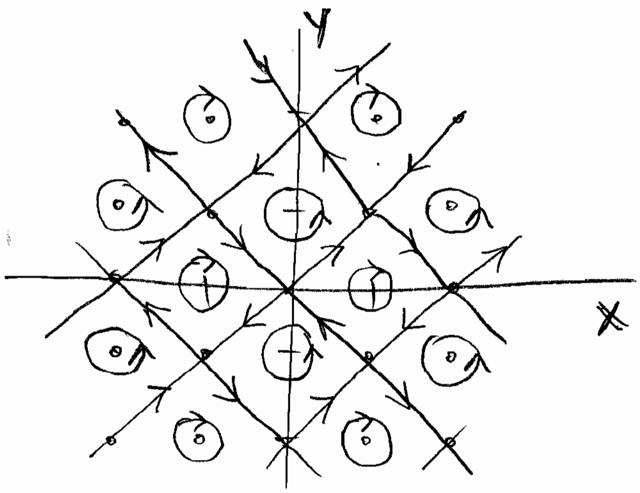
d) $(x^*, y^*) = (2\pi m, 2\pi n)$: $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_\pm = \pm 1 \to$ Saddle

$(x^*, y^*) = (\pi + 2\pi m, \pi + 2\pi n)$: $J = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda_\pm = \pm 1$

$\to$ saddle

$(x^*, y^*) = (2\pi m, \pi + 2\pi n)$: $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_\pm = \pm i \to$ center

$(x^*, y^*) = (\pi + 2\pi n, 2\pi m)$: $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda_\pm = \pm i \to$ Center



6.7.2

γ is meant to be the magnitude of the torque, $\gamma \geq 0$.

a) The equation can be rewritten as a system:

$$\begin{cases} \dot{\theta} = \nu \\ \dot{\nu} = \gamma - \sin\theta \end{cases}$$

fixed points occur at $(\arcsin(\gamma), 0)$

Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}$$

eigenvalues are $\lambda_{+/-} = \pm\sqrt{-\cos\theta}$

so that when $\gamma \neq 0$ there are two fixed points in an interval of length 2π, such that one has a positive cosine and the other has a negative cosine. $\cos\theta^* = \pm\sqrt{1-\gamma^2}$

If $\cos\theta^* < 0 \Rightarrow \lambda_+ = \sqrt{\cos\theta^*}$ and $\lambda_- = -\sqrt{-\cos\theta^*}$

both real, one positive and one negative, we have a saddle. The other fixed point has eigenvalues

$$\lambda_+ = +i\sqrt{|\cos\theta^*|} \quad \lambda_- = -i\sqrt{|\cos\theta^*|}$$

that is it's a center.

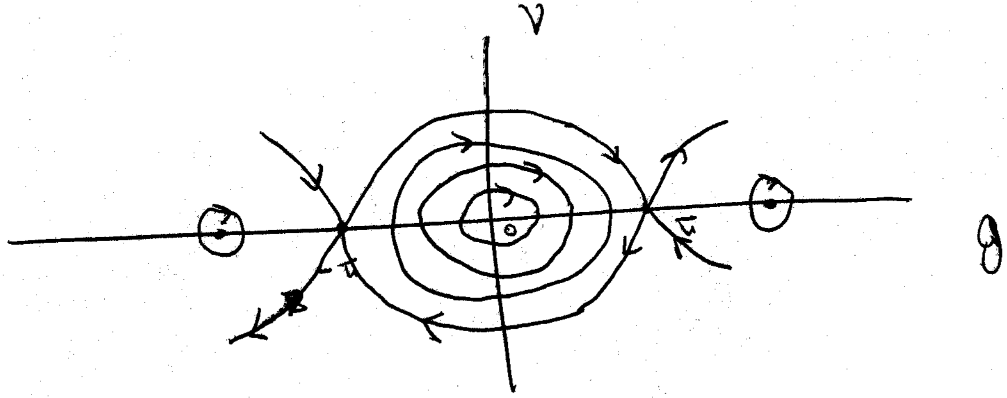
If $\gamma = 1$ there is only a fixed point per every interval of length 2π,

~~is~~ such that

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

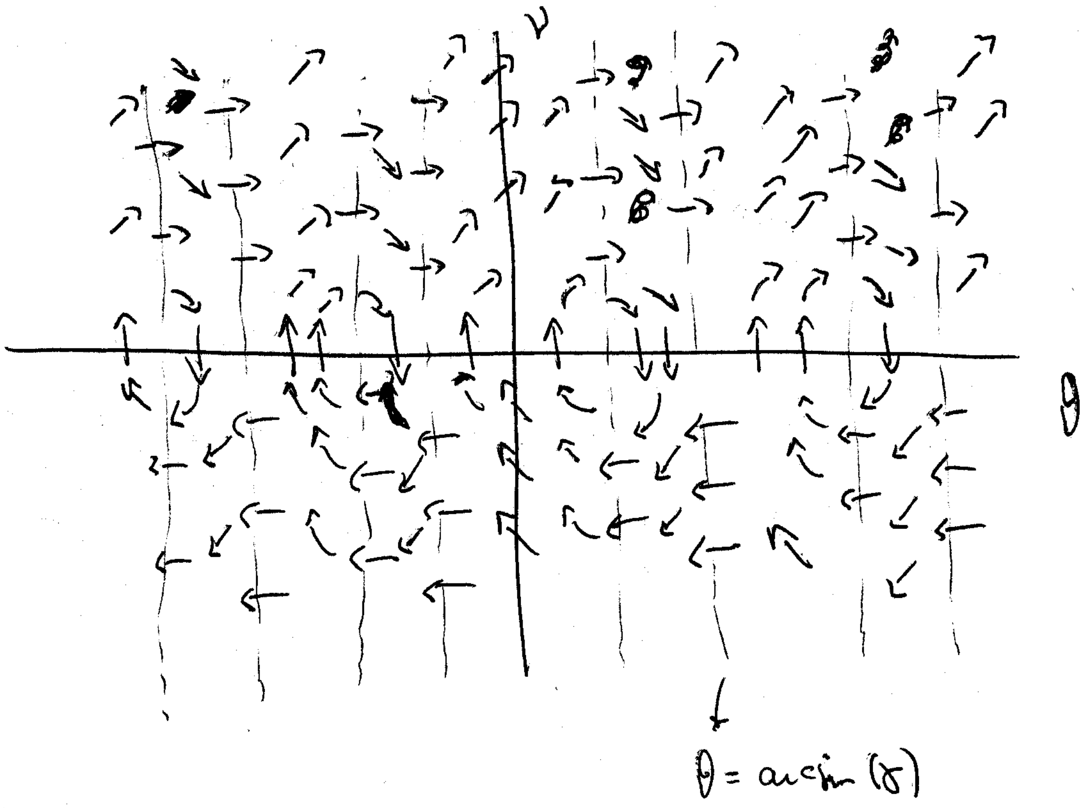
which implies that $\lambda_1 = \lambda_2 = 0$ ie marginal stability.

if $\gamma = 0$ there is no driving force and we can already sketch the phase portrait. (cf. page 170 of Strogatz's)

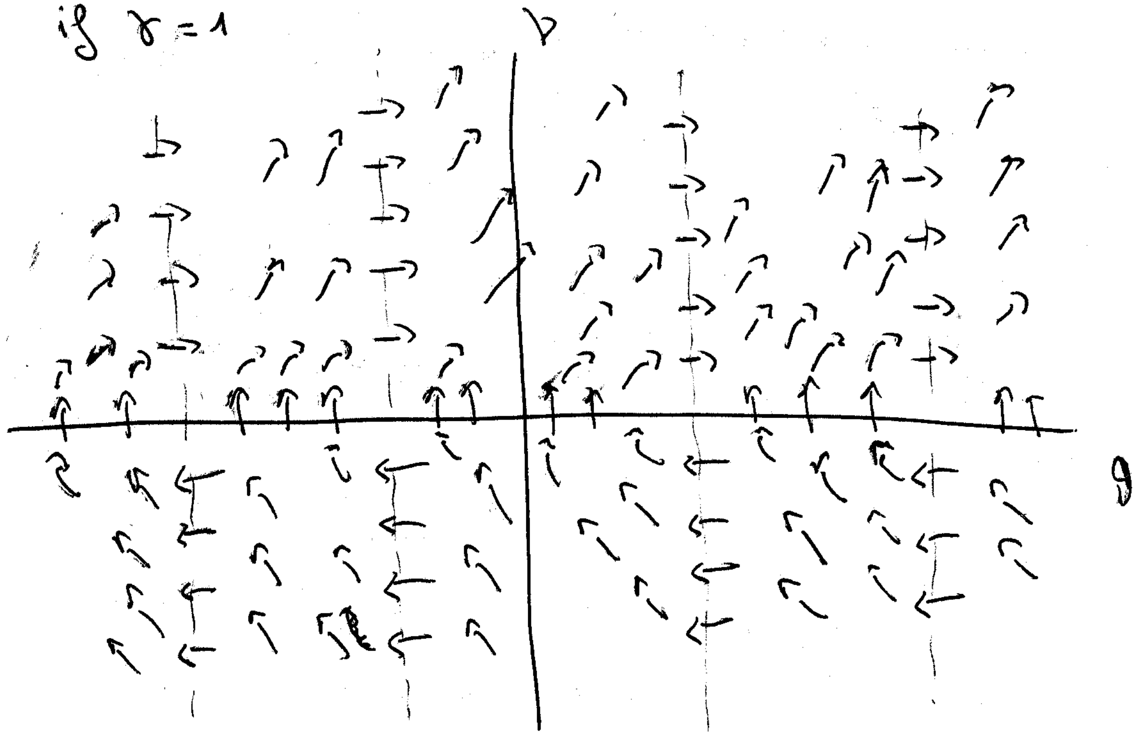


If $\gamma > 1$ there are no fixed points.

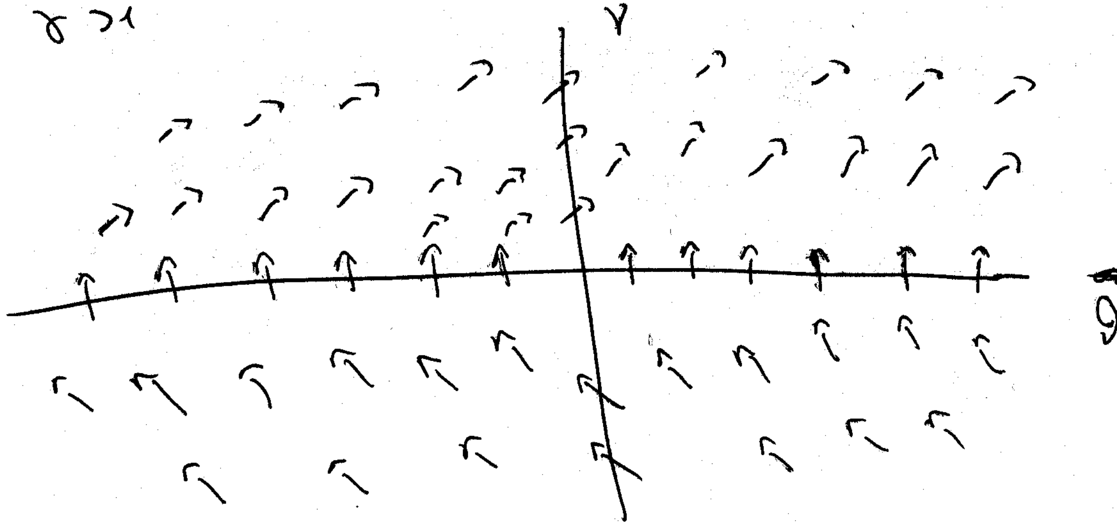
b) If $\gamma < 1$ nullclines ~~are~~ $v = 0$ and $\theta = \arcsin(\gamma)$



if $\gamma = 1$



if $\gamma > 1$



only nullcline $v = 0$

c) $\ddot{\theta} + \sin\theta = \gamma$

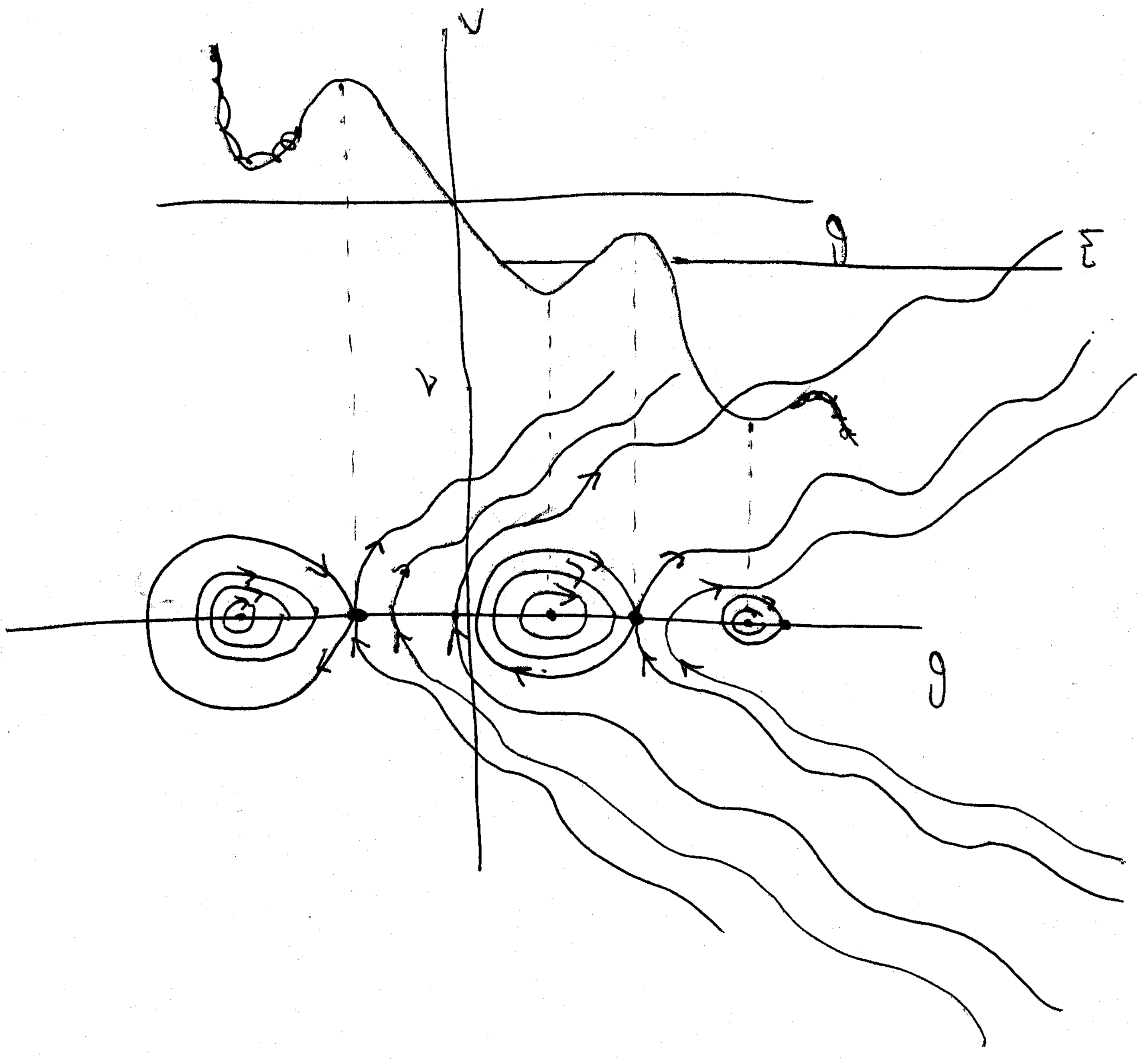
multiply everything by $\dot{\theta}$ and get

$$\dot{\theta}\ddot{\theta} + \dot{\theta}\sin\theta = \gamma\dot{\theta} \iff \frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \left[\cos\theta + \gamma\theta\right]\right] = 0$$

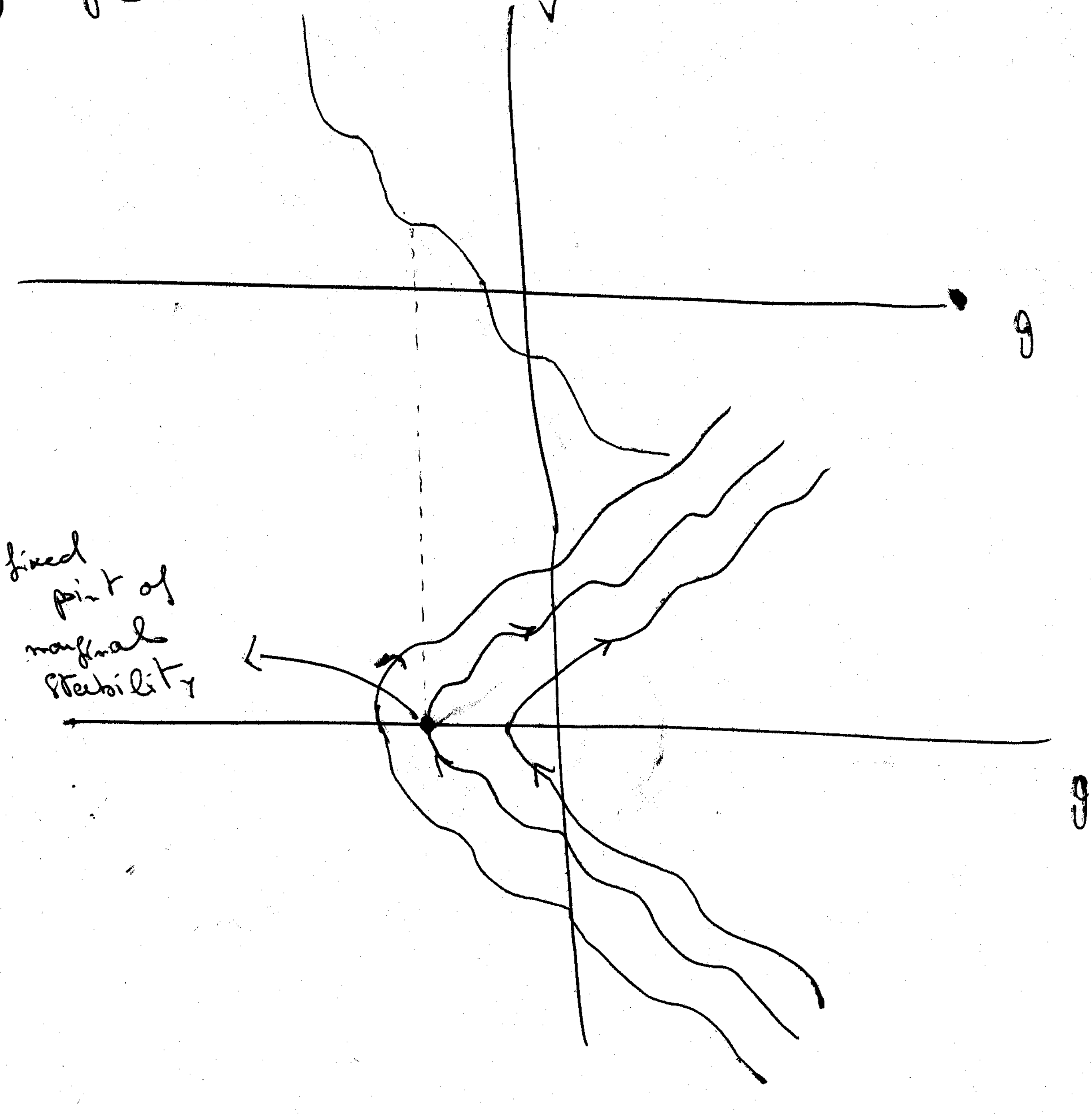
so that $\frac{1}{2}\dot{\theta} - (\cos\theta + \gamma\theta) =$ constant $\to$ it's conservative

When $t \to -t$ the second derivative $\ddot{\theta}$ stays the same, just like all the rest, so it's also time reversible.

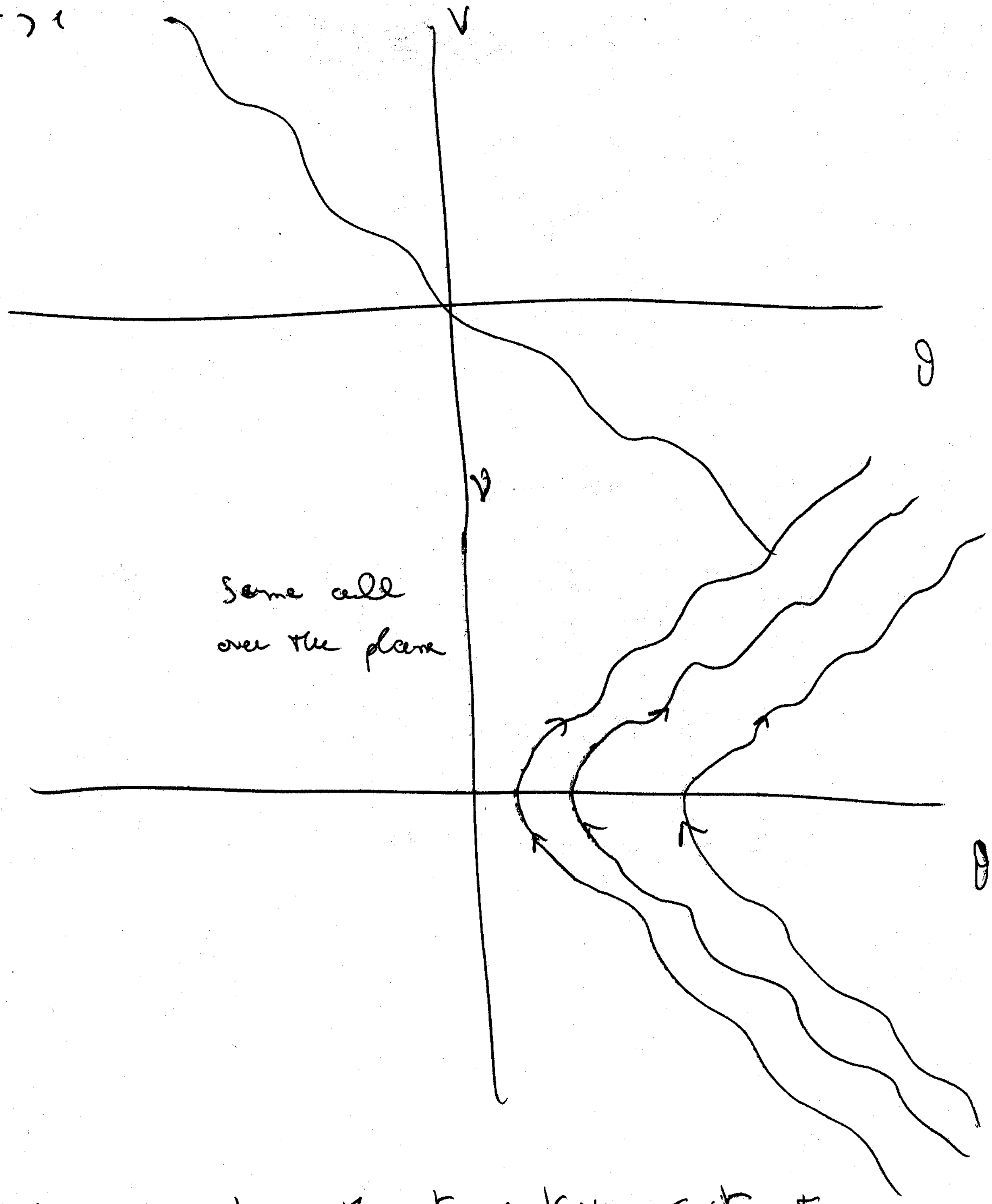
d) From c) we define the potential $V = -\gamma\theta - \cos\theta$ and we use its graph to sketch the phase portrait of the system (cf. problem 6.5.10)



ii) $\gamma = 1$



iii) $\gamma > 1$



No fixed points, the trajectories just turn around after having reached a certain θ

e) Centers occur when $\cos\vartheta^* = -\sqrt{1-\gamma^2}$

($\gamma = \sin\vartheta^*$ and we want the cosine to be negative in order to have imaginary eigenvalues)

Therefore let's expand the equation of motion around that value:

$\ddot{\vartheta} + \sin\vartheta = \gamma$ becomes

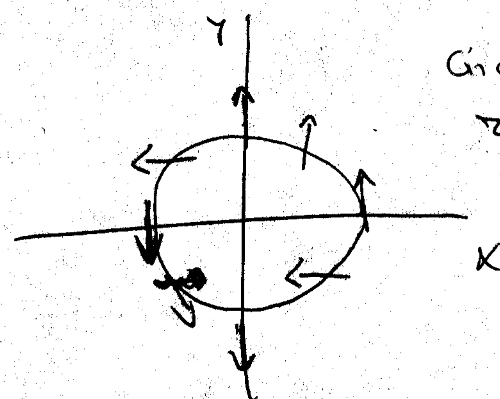
$$\ddot{\vartheta} + \gamma - \sqrt{1-\gamma^2}\left(\vartheta - \arcsin\gamma\right) + O\left(\vartheta^2\right) = \gamma$$

$$\Leftrightarrow \quad \ddot{\vartheta} = \sqrt{1-\gamma^2}\left(\vartheta - \arcsin\gamma\right)$$

it's ~~an oscillat~~ a linear ~~oscillat~~ harmonic oscillator of frequency $\omega = \left(1-\gamma^2\right)^{1/4}$

6.8.5 HW 8

$$\begin{cases} \dot{x} = xy \\ \dot{y} = x+y \end{cases}$$



Circle of radius 1

$f(1,0) = (0,1)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}, \sqrt{2}\right)$

$f(0,1) = (0,1)$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{2}, 0\right)$

$f(-1,0) = (0,-1)$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}, -\sqrt{2}\right)$

$f(0,-1) = (0,-1)$

$f\left(+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{2}, 0\right)$

From $(1,0)$ to $(-1,0)$ the field changes by $+\pi$, from $(-1,0)$ to $(0,-1)$ the field direction doesn't change. From $(0,-1)$ to $(1,0)$ there is a $-\pi$ change.

Hence $[I_c(0,0)] = \frac{-\pi + \pi}{2\pi} = 0$

6.8.7

$$\begin{cases} \dot{x} = x(4-y-x^2) \\ \dot{y} = y(x-1) \end{cases}$$

fixed points at

$(0,0)$, $(2,0)$; $(-2,0)$

$(1,3)$

$$J = \begin{pmatrix} 4-y-3x^2 & -x \\ y & x-1 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

so that the origin is a saddle point

(index -1)

$$J(-2,0) = \begin{pmatrix} -8 & 2 \\ 0 & -3 \end{pmatrix}$$

of eigenvalues $\lambda_1 = -8$, $\lambda_2 = -3$, therefore it is a stable node (index +1).

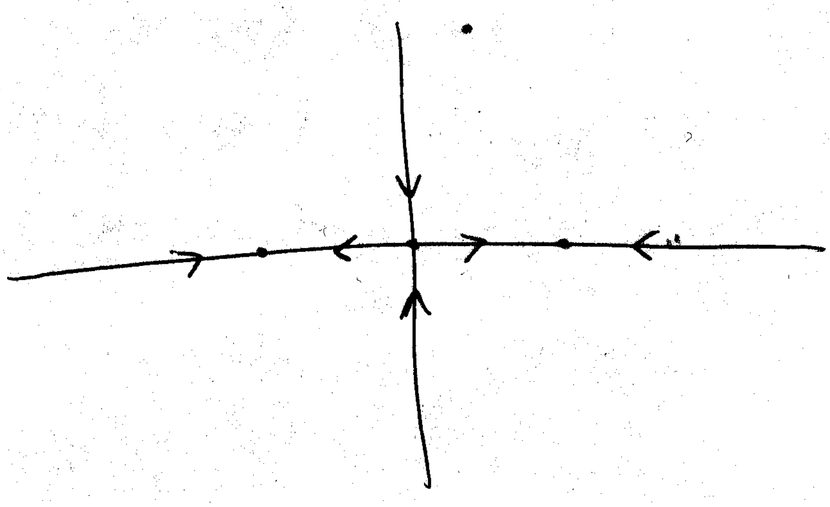
$$J(2,0) = \begin{pmatrix} -8 & -2 \\ 0 & 1 \end{pmatrix}$$

of eigenvalues $\lambda_1 = -8$, $\lambda_2 = 1$, that is a saddle point (index -1)

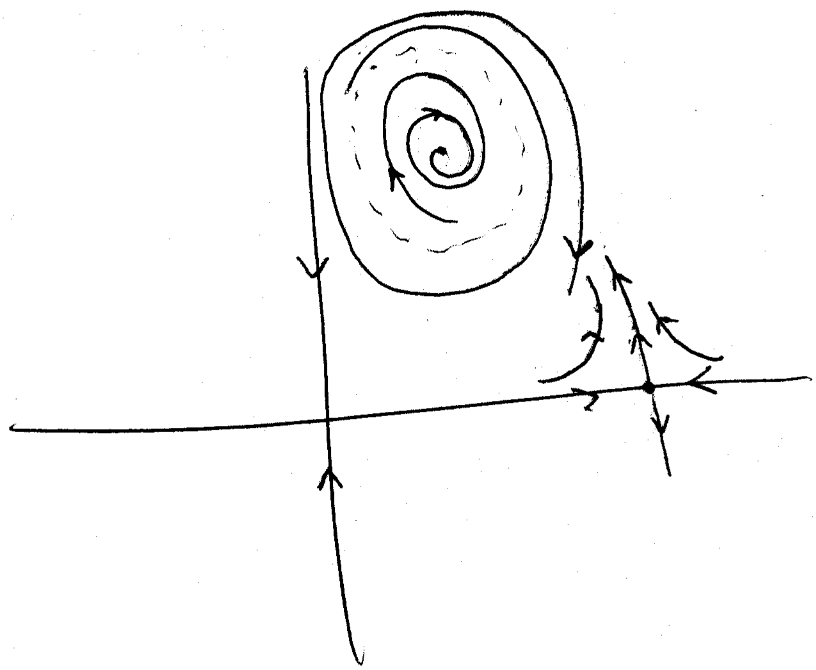
$$J(1,3) = \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix}$$

eigenvalues $\lambda = -1 \pm \sqrt{2}\, i$

that is a stable spiral (index +1).



the x-axis contains straight-line trajectories, so that all the closed orbits crossing the x-axis can be ruled out. Therefore the only option is a closed orbit encircling the fixed point (1,3), which is unstable spiral. ~~the only possible~~ ~~eq~~ see



~~the only~~ If there were such a closed orbit, it would be an unstable limit cycle (see ch. 7), so that there would be ~~spirals~~ orbits spiraling out of it, but they would ~~also~~ cross the trajectories going out of the saddle-point at (2,0), which is impossible. Therefore there are no closed orbits in this system.