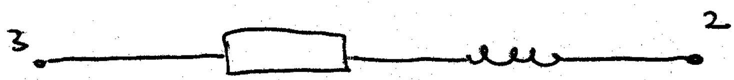
7.1.6

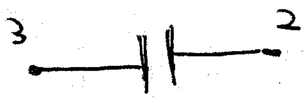
a) let us evaluate the voltage difference between points V_3 and V_2

$$V_3 - V_2 \equiv V = L\frac{dI}{dt} + f(I)$$



a one side, for Kirchoff law.

On the other side,



$$V_3 - V_2 = -\frac{Q}{C} \quad \Rightarrow \quad \dot{V} = -\frac{dQ}{dt} C = -IC$$

b) let now $x = L^{1/2} I$, $w = C^{1/2} V$, $\tau = (LC)^{-1/2} t$,

$f(L^{1/2} x) = F(x)$

$\Rightarrow$ first equation becomes

$$C^{-1/2} w = \frac{L}{\sqrt{LC}} (\sqrt{L})^{-1} \frac{dx}{d\tau} + f(I)$$

$$\Rightarrow \quad w - f(I)\sqrt{C} = \frac{dx}{d\tau}, \qquad f(I) = F(x)$$

$$\Rightarrow \quad w - \sqrt{C}\, F(x) = \frac{dx}{d\tau}$$

$$\boxed{\mu = \sqrt{C}}$$

the second equation is

$$\frac{C^{-1/2}}{\sqrt{LC}}\frac{d\bar{w}}{d\tau} = -\frac{L^{-1/2}}{C}x \iff \frac{dw}{d\tau} = -x$$

7.2.9

a) $\dot{x} = y + x^2 y$, if that is $-\frac{\partial V}{\partial x}$

then $V = \cancel{\exp} - \int f(x,y) + V_1(y)$

$= -xy - \frac{x^3}{3}y + V_1(y)$

but then $-\frac{\partial V}{\partial y} = +x + \frac{x^3}{3} + V_1'(y) \neq -x + 2xy$

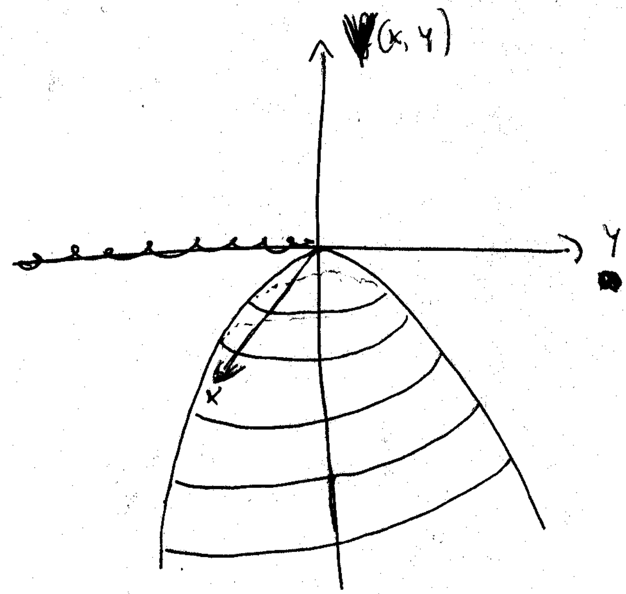
for no (x,y). therefore that is not a gradient system.

b) $\dot{x} = 2x = -\frac{\partial V}{\partial x} \Rightarrow V = -x^2 + V_1(y)$

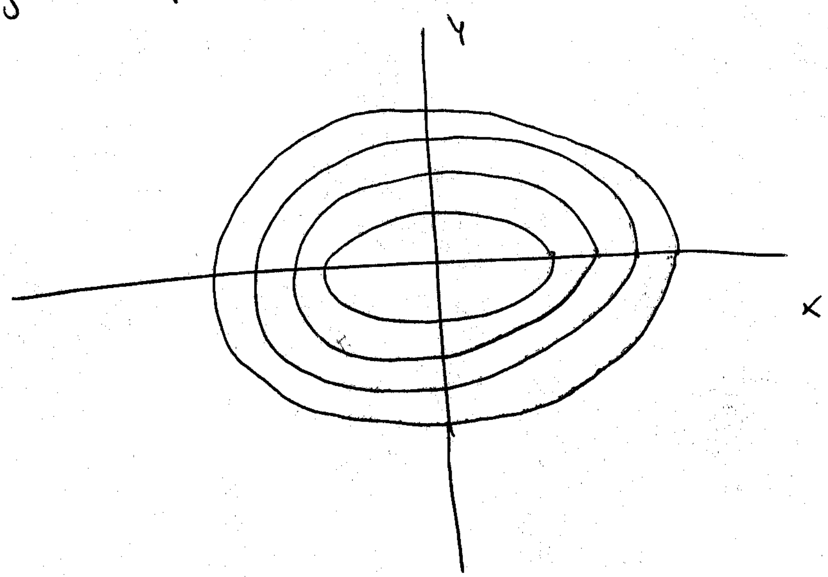
Now $-\frac{\partial V}{\partial y} = -\frac{\partial \bar{V}_1(y)}{\partial y} = 8y$ if $\bar{V}_1(y) = -4y^2$

so that $\bar{V} = -x^2 - 4y^2$

$V = -x^2 - 4y^2$ is a three-dimensional paraboloid.



Its x-y sections, that is the equipotentials, is obtained by cutting the paraboloid with a plane $z = E$.



$$-x^2 - 4y^2 = E \iff \frac{x^2}{(-4E)} + \frac{y^2}{(-E)} = +1$$

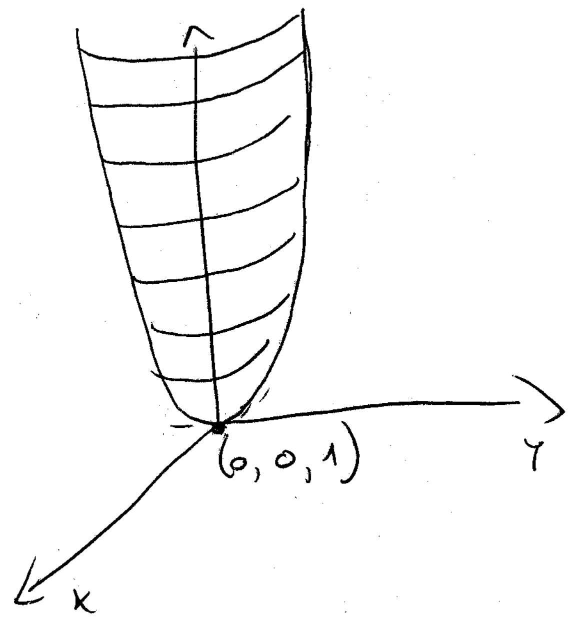
that is a family of ellipses of axes $\sqrt{-4E}$ and $\sqrt{-E}$

, $E \leq 0$.

c) $\dot{x} = -2x\, e^{x^2+y^2} = -\dfrac{\partial V}{\partial x}$

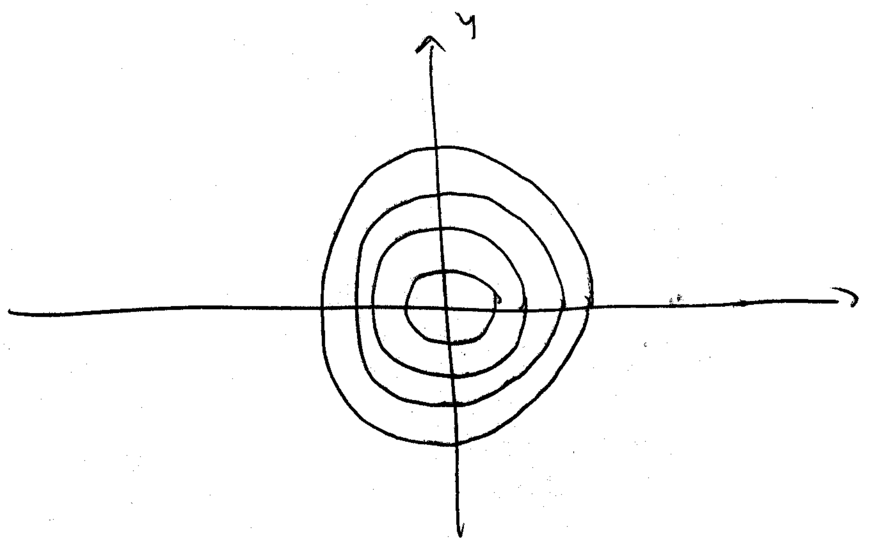
$\dot{y} = -2y\, e^{x^2+y^2} = -\dfrac{\partial V}{\partial y}$

then it's easy to guess that $V = e^{x^2+y^2}$



The x-y sections (i.e. the equipotentials) are obtained by setting $V = \xi$, that is $e^{x^2+y^2} = \xi$, $\xi \geq 1$

$\Rightarrow \quad x^2 + y^2 = \ln \xi$ that is a family of circles of radius $\sqrt{\ln \xi}$



7.3.9 HW #9

a) Assume $r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$

$$\Rightarrow \frac{dr}{d\theta} = \mu r_1' = (1+\mu r_1)\left(1-(1+\mu r_1)^2\right) + \mu(1+\mu r_1)\cos\theta$$

$$\Leftrightarrow \mu r_1' = (1+\mu r_1)(-2\mu r_1) + \mu\cos\theta$$

$$\Leftrightarrow \mu r_1' = -2\mu r_1 + \mu\cos\theta$$

$$\Leftrightarrow \frac{dr_1}{d\theta} = -2r_1 + \cos\theta \qquad (*)$$

which is an inhomogeneous first order differential equation. The way to ~~look for~~ solve it is to look for a general solution of the homogeneous equation

$$\frac{dr_1}{d\theta} = -2r_1$$

$$\Rightarrow r_1 = C(\theta)e^{-2\theta}$$

notice that $C = C(\theta)$. Now let's plug that into $(*)$:

$$\frac{dC}{d\theta}e^{-2\theta} - 2Ce^{-2\theta} = -2Ce^{-2\theta} + \cos\theta \quad\Rightarrow$$

$$\frac{dC}{d\theta} = e^{2\theta}\cos\theta \quad\Rightarrow C = \frac{2}{5}e^{2\theta}\cos\theta + \frac{1}{5}e^{2\theta}\sin\theta$$

So that

$$r = 1 + \mu\left(\frac{2}{5}\cos\theta + \frac{1}{5}\sin\theta\right) + O\left(\mu^2\right)$$

b)

$$\frac{dr}{d\theta} = -\mu\,\frac{2}{5}\sin\theta + \frac{1}{5}\cos\theta = 0$$

$$\Leftrightarrow \quad \operatorname{tg}\theta = 1/2$$

which means either $\theta \cong 26^\circ$ and $\begin{cases} \cos\theta \simeq 0.89 \\ \sin\theta \simeq 0.44 \end{cases}$

or $\theta^\circ \cong 206^\circ$ and $\begin{cases} \cos\theta \cong 0.89 \\ \sin\theta \simeq 0.44 \end{cases}$ since

$r \cong 1 + \mu\left(\frac{2}{5}\cos\theta + \frac{1}{5}\sin\theta\right)$ we can infer that

$\theta = 26^\circ$ gives the maximum radius and the other gives the minimum.

Now $r(26^\circ) \cong 1 + \mu\left(\frac{2}{5}\cdot 0.89 + \frac{1}{5}\cdot 0.44\right)$

$$\simeq 1 + 0.44\mu < \sqrt{1+\mu} \quad \Leftrightarrow$$

$$\cancel{1} + (0.44)^2\mu^2 + 0.88\mu < \cancel{1} + \mu$$

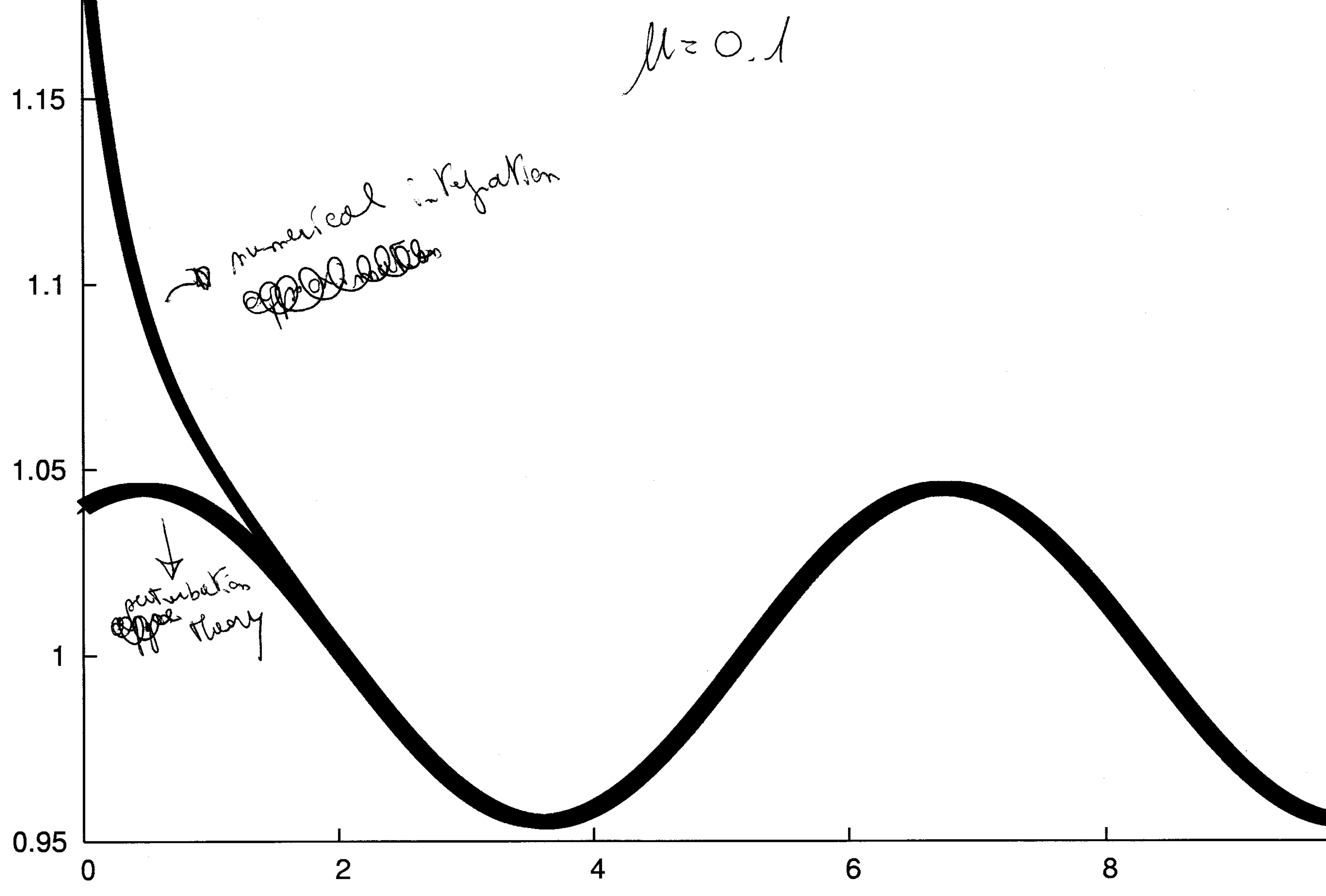
$$\Leftrightarrow \quad (0.44)^2\mu^2 < 0.22\mu$$

which is true for $\mu \ll 1$.

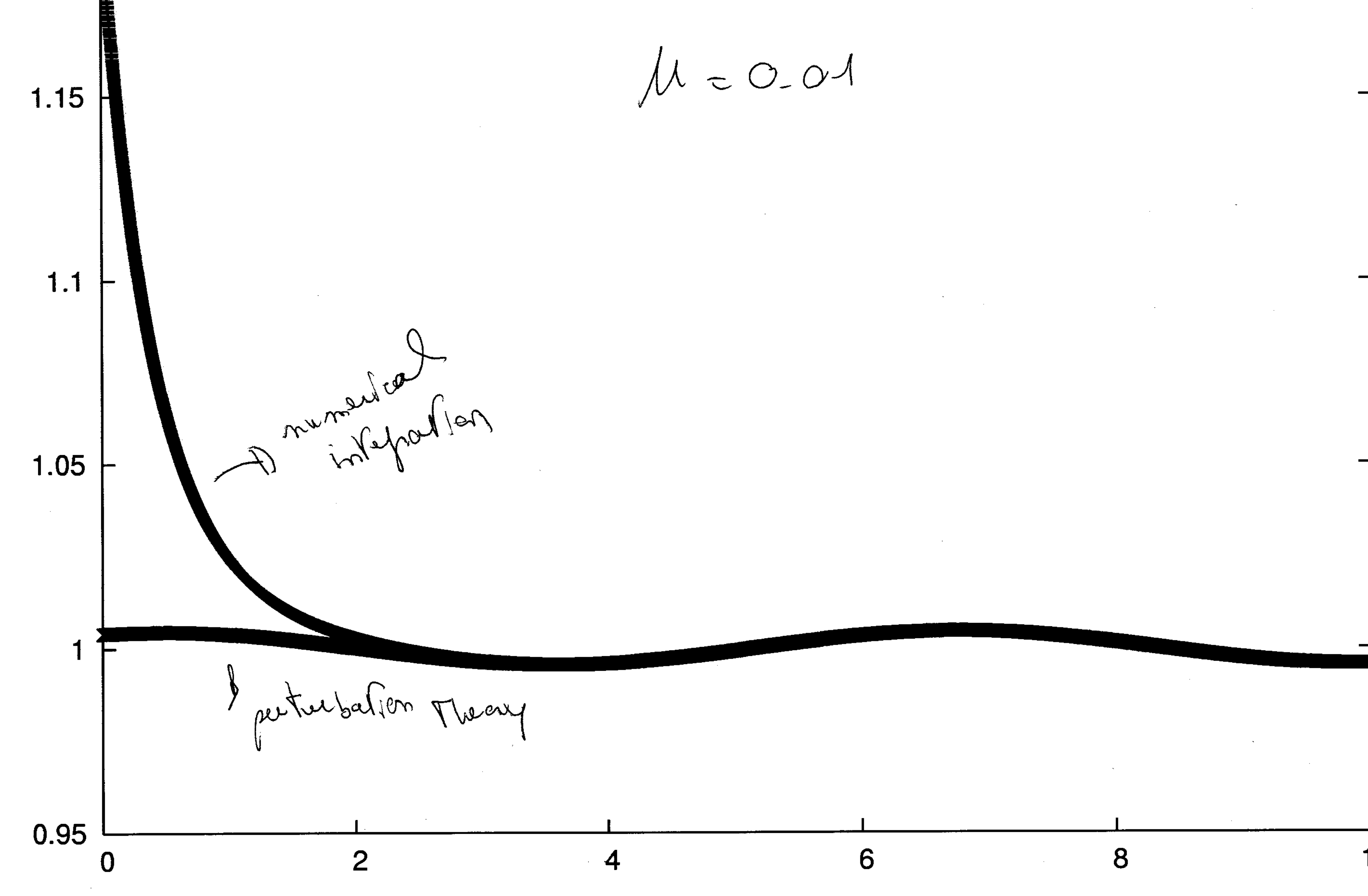
Analogously it's possible to prove that

$$r(206^\circ) > \sqrt{1-\mu} \quad \text{for } \mu \ll 1,$$

so that in general the radius of our orbit lies in the annulus $]\sqrt{1-\mu}, \sqrt{1+\mu}[$.



$\mu = 0.1$
numerical integration
perturbation theory
1.15
1.1
1.05
1
0.95
0
2
4
6
8



$\mu = 0.01$
numerical integration
perturbation theory
1.15
1.1
1.05
1
0.95
0
2
4
6
8

7.4.1 this is the Lienard System with

$f(x) = \mu(x^2-1)$, $g(x) = \tanh(x)$.

Furthermore, $F(x) = \int_0^x f(u)du = \mu\left(\frac{x^3}{3} - x\right) = \frac{\mu}{3} x (x-\sqrt{3})(x+\sqrt{3})$

It is easy to check that all conditions of the Lienard Theorem are satisfied (for $\mu > 0$), so the system possesses a unique stable limit cycle.

For $\mu < 0$ the last condition of the Theorem is not satisfied ($F(x) > 0$, for $0 < x < \sqrt{3}$), so the existence and uniqueness of the limit cycle cannot be established.